Probabilistic Linear Logic Programming

with an Application to Bayesian Network Computations

Roberto Maieli

Department of Mathematics and Physics
"Roma TRE" University

(joint work with **Matteo Acclavio**, University of Southern Denmark)

FLOPS - 2026, May 26

this talk in one slide

- **probLO**: *probabilistic Linear Objects* is a LP language:
  - based on *Linear Logic* (Girard, 1987)
  - extension of Andreoli-Pareschi's *LO language*
  - endowed with *probabilities* ($p \in [0, 1]$)
- The main features are:
  - **multi-set** of literals for the head and the body of (extended) Prolog-like methods to deal with complex network structures (not simply trees)

$$\underbrace{p}_{\text{probability}} :: [\underbrace{h_1, \ldots, h_n}_{\text{head}}] :- [\underbrace{b_1, \ldots, b_m}_{\text{body}}]$$

  - the ability to **compute internal numerical probability computations** without relying on external semantic interpretation.
- **probLO** is powerful enough:
  1. to represent **Bayesian Networks**
     (their integration within logic programming frameworks remains a nontrivial challenge, mainly due to their complex structure)
  2. to compute probabilistic inferences as **joint** or **marginal probabilities**.

preliminaries on probabilities

- **probability distribution** for **random (Boolean) variable** X

$$\mathsf{Pr}:\underbrace{\mathbf{X}}_{\textit{events space}}\rightarrow[0,1]\subset\mathbb{R}\quad\text{s.t.}\quad\mathsf{Pr}(\underbrace{\mathsf{X}=\mathsf{True}}_{\textit{event } x})+\mathsf{Pr}(\underbrace{\mathsf{X}=\mathsf{False}}_{\textit{event } \overline{x}})=1$$

- **joint probability** of the events $x_1,\ldots,x_n$ s.t. $x_i\in\{\mathsf{X}_i=\mathsf{True},\mathsf{X}_i=\mathsf{False}\}$

$$\mathsf{Pr}(\underbrace{x_1,...,x_n}_{\textit{events}})$$

- **marginal probability** of events $y_1,\ldots,y_m$ regardless of variables $\mathsf{X}_1,...,\mathsf{X}_n$

$$\mathsf{Pr}(y_1,\ldots,y_m)=\sum_{x_1\in\mathbf{X}_1,\ldots,x_n\in\mathbf{X}_n}\mathsf{Pr}(x_1,\ldots,x_n,y_1,\ldots,y_m)$$

- **conditional probability** of $x_1,...,x_n$ given the **evidences** $y_1,\ldots,y_m$

$$\mathsf{Pr}(x_1,\ldots,x_n\mid\underbrace{y_1,\ldots,y_m}_{\textit{evidences}})=\frac{\mathsf{Pr}(x_1,\ldots,x_n,y_1,\ldots,y_m)}{\mathsf{Pr}(y_1,\ldots,y_m)}$$

- **factorization** (*chain rule*):

$$\mathsf{Pr}(x_1,\ldots,x_n)=\mathsf{Pr}(x_1\mid x_2,\ldots,x_n)\mathsf{Pr}(x_2\mid x_3,\ldots,x_n)\cdots\mathsf{Pr}(x_{n-1}\mid x_n)\mathsf{Pr}(x_n)$$

Bayesian networks (BN)

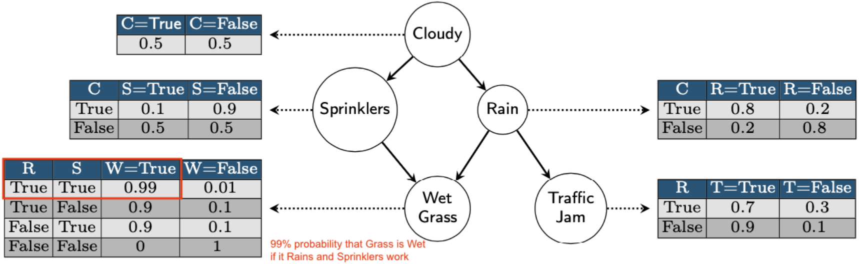
(Bool) Bayesian Network $\mathcal{B} = \langle V_{\mathcal{B}}, \rightharpoonup_{\mathcal{B}}, \mathcal{P}_{\mathcal{B}} \rangle$ is a probabilistic graphical model

- $\langle V_{\mathcal{B}}, \rightharpoonup_{\mathcal{B}} \rangle$ is a DAG :
  - $V_{\mathcal{B}}$ is a set of random (Boolean) variables,
  - $\rightharpoonup_{\mathcal{B}} \subset V_{\mathcal{B}} \times V_{\mathcal{B}}$ is a set of edges
- $\mathcal{P}_{\mathcal{B}} : \mathrm{X} \in V_{\mathcal{B}} \mapsto CTP_{\mathrm{X}}$ (the **conditional probability table of** X)

a *table row* :

$\Pr(\mathrm{X} = \mathrm{True} \mid y_1, \ldots, y_n)$	$\Pr(\mathrm{X} = \mathrm{False} \mid y_1, \ldots, y_n)$

$$\underbrace{\Pr(\mathrm{X} \mid \textbf{Parents}(\mathrm{X}))}_{\text{the conditional probability of X, given evidences for } \textit{Parents}(\mathrm{X})}$$



C=True	C=False
0.5	0.5

C	S=True	S=False
True	0.1	0.9
False	0.5	0.5

C	R=True	R=False
True	0.8	0.2
False	0.2	0.8

R	S	W=True	W=False
True	True	0.99	0.01
True	False	0.9	0.1
False	True	0.9	0.1
False	False	0	1

R	T=True	T=False
True	0.7	0.3
False	0.9	0.1

99% probability that Grass is Wet
if it Rains and Sprinklers work

Bayesian networks: queries computation

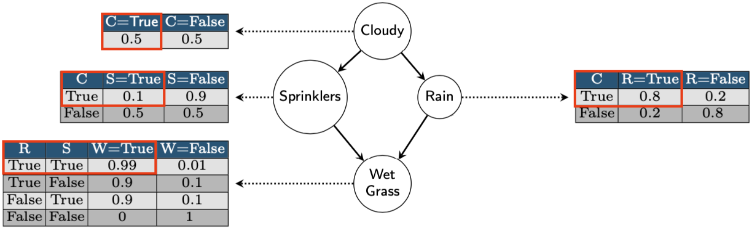
A BN represents joint probability distributions in a compact way$^{(*)}$

Theorem (*Factorization of Bayesian Networks*)

Let $\mathcal{B} = \langle V, \rightharpoonup, \mathcal{P} \rangle$ *be a BN then* $\Pr(X_1, \ldots, X_n) = \prod_{X_i \in V} \Pr(X_i \mid \textbf{Parents}(X_i))$.

Example: assume events: $\underbrace{C = \text{True}}_{c}, \underbrace{S = \text{True}}_{s}, \underbrace{R = \text{True}}_{r}, \underbrace{W = \text{True}}_{w}$

$$\text{compute:} \quad \Pr(c, s, r, w) = \underbrace{\Pr(w \mid r, s)}_{0.99} \times \underbrace{\Pr(s \mid c)}_{0.1} \times \underbrace{\Pr(r \mid c)}_{0.8} \times \underbrace{\Pr(c)}_{0.5} = 0.0396$$



C=True	C=False
0.5	0.5

C	S=True	S=False
True	0.1	0.9
False	0.5	0.5

C	R=True	R=False
True	0.8	0.2
False	0.2	0.8

R	S	W=True	W=False
True	True	0.99	0.01
True	False	0.9	0.1
False	True	0.9	0.1
False	False	0	1

(∗) Only 9 conditional probabilities instead of $16 = 2^4$ joint probabilities!

a naive implementation of conditional probabilities in Prolog? 1/3

a conditional probability as a Prolog method with a probability $p \in [0, 1]$:

$$\underbrace{\Pr(x \mid y_1, \ldots, y_m) = p}_{\text{conditional probability}} \quad \rightsquigarrow \quad \underbrace{p}_{\text{probability}} :: [\underbrace{\mathtt{w}_{\mathtt{x}}}_{\text{head}}] :\!- [\underbrace{\mathtt{w}_{\mathtt{y}_1}, \ldots, \mathtt{w}_{\mathtt{y}_\mathtt{m}}}_{\text{body}}]$$

where :

- $\mathtt{w}_{\mathtt{x}}, \mathtt{w}_{\mathtt{y}_\mathtt{i}}$ are literals

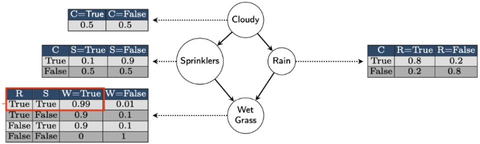
$$\mathtt{w}_{\mathrm{x}} = \begin{cases} \mathit{true}_{\mathrm{X}} & \text{if } x \text{ is the event "X = True"} \\ \mathit{false}_{\mathrm{X}} & \text{if } x \text{ is the event "X = False"} \end{cases}$$

$$\mathtt{w}_{\mathrm{y_i}} = \begin{cases} \mathit{true}_{\mathrm{Y}_i} & \text{if } y_i \text{ is the event "Y}_i\text{ = True"} \\ \mathit{false}_{\mathrm{Y}_i} & \text{if } y_i \text{ is the event "Y}_i\text{ = False"} \end{cases}$$

- comma "," is interpreted by the **(additive) conjunction** $\wedge$

$$\underbrace{p}_{\text{probability}} :: [\underbrace{\mathtt{w}_{\mathtt{x}}}_{\text{head}}] :\!- [\underbrace{\mathtt{w}_{\mathtt{y}_1} \wedge \ldots \wedge \mathtt{w}_{\mathtt{y}_\mathtt{m}}}_{\text{body}}]$$

a naive implementation of a Bayesian Network in Prolog? 2/3



C=True	C=False
0.5	0.5

C	S=True	S=False
True	0.1	0.9
False	0.5	0.5

C	R=True	R=False
True	0.8	0.2
False	0.2	0.8

R	S	W=True	W=False
True	True	0.99	0.01
True	False	0.9	0.1
False	True	0.9	0.1
False	False	0	1

$0.50 :: [\mathtt{true_C}\ :-\ .]$	$0.50 :: [\mathtt{false_C}\ :-\ .]$
$0.10 :: [\mathtt{true_S}\ :-\ \mathtt{true_C}]$	$0.90 :: [\mathtt{false_S}\ :-\ \mathtt{true_C}]$
$0.10 :: [\mathtt{true_S}\ :-\ \mathtt{false_C}]$	$0.90 :: [\mathtt{false_S}\ :-\ \mathtt{false_C}]$
$0.80 :: [\mathtt{true_R}\ :-\ \mathtt{true_C}]$	$0.20 :: [\mathtt{false_R}\ :-\ \mathtt{true_C}]$
$0.20 :: [\mathtt{true_R}\ :-\ \mathtt{false_C}]$	$0.80 :: [\mathtt{false_R}\ :-\ \mathtt{false_C}]$
$\boxed{0.99 :: [\mathtt{true_W}\ :-\ \mathtt{true_R}, \mathtt{true_S}]}$	$0.01 :: [\mathtt{false_W}\ :-\ \mathtt{true_R}, \mathtt{true_S}]$
$0.90 :: [\mathtt{true_W}\ :-\ \mathtt{true_R}, \mathtt{false_S}]$	$0.10 :: [\mathtt{false_W}\ :-\ \mathtt{true_R}, \mathtt{false_S}]$
$0.90 :: [\mathtt{true_W}\ :-\ \mathtt{false_R}, \mathtt{true_S}]$	$0.10 :: [\mathtt{false_W}\ :-\ \mathtt{flase_R}, \mathtt{true_S}]$
$0.00 :: [\mathtt{true_W}\ :-\ \mathtt{false_R}, \mathtt{false_S}]$	$1.00 :: [\mathtt{false_W}\ :-\ \mathtt{flase_R}, \mathtt{false_S}]$

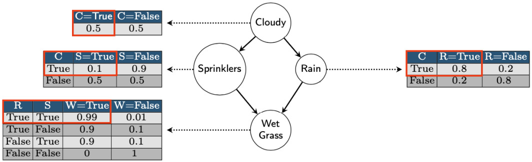
Resolution as Sequent Calc. inference:

$$\frac{\dfrac{\vdash \mathcal{P}; \mathrm{b}_1 \qquad \vdash \mathcal{P}; \mathrm{b}_i \qquad \vdash \mathcal{P}; \mathrm{b}_m}{\vdash \mathcal{P}; \mathrm{b}_1 \wedge \ldots \wedge \mathrm{b}_m}\ \wedge}{\vdash\ \underbrace{\mathcal{P}, M}_{\texttt{program}}\ ;\ \underbrace{h}_{\texttt{goal}}}\ \mathtt{M} = ([\mathtt{h}] : -[\mathtt{b}_1, \ldots, \mathtt{b}_m])^{\texttt{method}}$$

probabilistic inference:

$$\frac{\mathrm{q}_1 :: \mathcal{P}; \mathrm{h}_1 \qquad \mathrm{q}_i :: \mathcal{P}; \mathrm{h}_i \qquad \mathrm{q}_m :: \mathcal{P}; \mathrm{h}_m}{\underbrace{q_1 \cdots q_m \cdot p}_{\texttt{probability}} :: \underbrace{\mathcal{P}, M}_{\texttt{program}}\ ;\ \underbrace{h}_{\texttt{goal}}}\ \mathtt{M} = (\mathtt{p} :: [\mathtt{h}] : -[\mathtt{b}_1, \ldots, \mathtt{b}_m])^{\texttt{method}}$$

a naive implementation of a Bayesian Network in Prolog? 3/3



C=True	C=False
0.5	0.5

C	S=True	S=False
True	0.1	0.9
False	0.5	0.5

C	R=True	R=False
True	0.8	0.2
False	0.2	0.8

R	S	W=True	W=False
True	True	0.99	0.01
True	False	0.9	0.1
False	True	0.9	0.1
False	False	0	1

$$\Pr(c, s, r, w) = \underbrace{\Pr(w \mid r, s)}_{0.99} \times \underbrace{\Pr(s \mid c)}_{0.10} \times \underbrace{\Pr(r \mid c)}_{0.80} \times \underbrace{\Pr(c)}_{0.50} = 0.0396; \qquad \text{linear computation}$$

program $\mathcal{P}$:

$0.50 :: [\mathtt{true}_\mathtt{C}\ : - .]$	$0.50 :: [\mathtt{false}_\mathtt{C}\ : - .]$
$0.10 :: [\mathtt{true}_\mathtt{S}\ : - \mathtt{true}_\mathtt{C}]$ $0.10 :: [\mathtt{true}_\mathtt{S}\ : - \mathtt{false}_\mathtt{C}]$	$0.90 :: [\mathtt{false}_\mathtt{S}\ : - \mathtt{true}_\mathtt{C}]$ $0.90 :: [\mathtt{false}_\mathtt{S}\ : - \mathtt{false}_\mathtt{C}]$
$0.80 :: [\mathtt{true}_\mathtt{R}\ : - \mathtt{true}_\mathtt{C}]$ $0.20 :: [\mathtt{true}_\mathtt{R}\ : - \mathtt{false}_\mathtt{C}]$	$0.20 :: [\mathtt{false}_\mathtt{R}\ : - \mathtt{true}_\mathtt{C}]$ $0.80 :: [\mathtt{false}_\mathtt{R}\ : - \mathtt{false}_\mathtt{C}]$
$\mathbf{0.99} :: [\mathbf{true}_\mathtt{W}\ : - \mathbf{true}_\mathtt{R}, \mathbf{true}_\mathtt{S}]$ $0.90 :: [\mathtt{true}_\mathtt{W}\ : - \mathtt{true}_\mathtt{R}, \mathtt{false}_\mathtt{S}]$ $0.90 :: [\mathtt{true}_\mathtt{W}\ : - \mathtt{false}_\mathtt{R}, \mathtt{true}_\mathtt{S}]$ $0.00 :: [\mathtt{true}_\mathtt{W}\ : - \mathtt{false}_\mathtt{R}, \mathtt{false}_\mathtt{S}]$	$0.01 :: [\mathtt{false}_\mathtt{W}\ : - \mathtt{true}_\mathtt{R}, \mathtt{true}_\mathtt{S}]$ $0.10 :: [\mathtt{false}_\mathtt{W}\ : - \mathtt{true}_\mathtt{R}, \mathtt{false}_\mathtt{S}]$ $0.10 :: [\mathtt{false}_\mathtt{W}\ : - \mathtt{flase}_\mathtt{R}, \mathtt{true}_\mathtt{S}]$ $1.00 :: [\mathtt{false}_\mathtt{W}\ : - \mathtt{flase}_\mathtt{R}, \mathtt{false}_\mathtt{S}]$

$$\frac{\dfrac{\dfrac{}{0.50 :: \mathtt{true}_\mathtt{C}}\ 0.50 :: [\mathtt{true}_\mathtt{C} : -.]}{0.40 = (0.50 \times 0.80) :: \mathtt{true}_\mathtt{R}}\ 0.80 :: [\mathtt{true}_\mathtt{R} : -\mathtt{true}_\mathtt{C}] \qquad \dfrac{\dfrac{}{0.50 :: \mathtt{true}_\mathtt{C}}\ 0.50 :: [\mathtt{true}_\mathtt{C} : -.]}{0.05 = (0.50 \times 0.10) :: \mathtt{true}_\mathtt{S}}\ 0.10 :: [\mathtt{true}_\mathtt{S} : -\mathtt{true}_\mathtt{C}]}{\vdash \underbrace{0.40 \times 0.05 \times 0.99}_{0.0198 \neq 0.0396\ \ \mathtt{non-linear\ computation\,!}} :: \mathtt{true}_\mathtt{W}}\ 0.99 :: [\mathtt{true}_\mathtt{W} : -\mathtt{true}_\mathtt{R}, \mathtt{true}_\mathtt{S}]$$

our solution: methods of LO (linear logic objects) (1/2)

$$\underbrace{\Pr(x_1, \ldots x_n \mid y_1, \ldots, y_m) = p}_{\text{conditional probability}} \quad \rightsquigarrow \quad \underbrace{p}_{\text{probability}} :: [\underbrace{\mathtt{w}_{\mathtt{x}_1}, \ldots, \mathtt{w}_{\mathtt{x}_n}}_{\text{head}}] :\!- [\underbrace{\mathtt{w}_{\mathtt{y}_1}, \ldots, \mathtt{w}_{\mathtt{y}_m}}_{\text{body}}]$$

- head $[\mathtt{w}_{\mathtt{x}_1}, \ldots, \mathtt{w}_{\mathtt{x}_n}]$ and body $[\mathtt{w}_{\mathtt{y}_1}, \ldots, \mathtt{w}_{\mathtt{y}_m}]$ are multiset of literals:

$$\mathtt{w}_{\mathrm{X_i}} = \begin{cases} \mathit{true}_{\mathrm{X}_i} & \text{if } x \text{ is the event } \mathrm{X}_i = \text{True} \\ \mathit{false}_{\mathrm{X}_i} & \text{if } x \text{ is the event } \mathrm{X}_i = \text{False} \end{cases} \qquad \mathtt{w}_{\mathrm{y_i}} = \begin{cases} \mathit{true}_{\mathrm{Y}_j} & \text{if } y_j \text{ is the event } \mathrm{Y}_j = \text{True} \\ \mathit{false}_{\mathrm{Y}_j} & \text{if } y_j \text{ is the event } \mathrm{Y}_j = \text{False} \end{cases}$$

- "," is interpreted by the **multiplicative disjunction $\parr$ of linear logic**

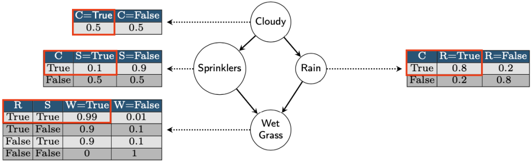
$$p :: [\mathtt{w}_{\mathtt{x}_1} \parr \cdots \parr \mathtt{w}_{\mathtt{x}_n}] \; : - \; [\mathtt{w}_{\mathtt{y}_1} \parr \cdots \parr \mathtt{w}_{\mathtt{y}_m}]$$

probabilistic inference:

$$\text{exp}\ \frac{q ::\ \mathrm{P}\ ;\ \Gamma, b_1, \ldots, b_m}{q \cdot p ::\ \mathrm{P}, \mathtt{M}\ ;\ \Gamma, h_1, \ldots, h_n}\, p :: [\mathtt{h}_1, \ldots, \mathtt{h}_\mathtt{n}] :\!- [\mathtt{b}_1, \ldots, \mathtt{b}_\mathtt{m}] = \mathtt{M}$$

prob. of conclusion $q \cdot p = q$ (prob. of premises) × p (prob. of applied method)

our solution: methods of LO (linear logic objects) (2/2)



C=True	C=False
0.5	0.5

C	S=True	S=False
True	0.1	0.9
False	0.5	0.5

C	R=True	R=False
True	0.8	0.2
False	0.2	0.8

R	S	W=True	W=False
True	True	0.99	0.01
True	False	0.9	0.1
False	True	0.9	0.1
False	False	0	1

$$\Pr(\mathrm{C}=\mathrm{True}, \mathrm{S}=\mathrm{True}, \mathrm{R}=\mathrm{True}, \mathrm{W}=\mathrm{True}) = \underbrace{\Pr(\mathrm{W} \mid \mathrm{R}, \mathrm{S})}_{0.99} \times \underbrace{\Pr(\mathrm{S} \mid \mathrm{C})}_{0.1} \times \underbrace{\Pr(\mathrm{R} \mid \mathrm{C})}_{0.8} \times \underbrace{\Pr(\mathrm{C})}_{0.5} = 0.0396$$

linear computation !

$0.50 :: [\mathtt{true}_\mathtt{C}, \mathtt{true}_\mathtt{C}\ :-\ .]$	$0.50 :: [\mathtt{false}_\mathtt{C}, \mathtt{false}_\mathtt{C}\ :-\ .]$
$0.10 :: [\mathtt{true}_\mathtt{S}\ :-\ \mathtt{true}_\mathtt{C}]$	$0.90 :: [\mathtt{false}_\mathtt{S}\ :-\ \mathtt{true}_\mathtt{C}]$
$0.10 :: [\mathtt{true}_\mathtt{S}\ :-\ \mathtt{false}_\mathtt{C}]$	$0.90 :: [\mathtt{false}_\mathtt{S}\ :-\ \mathtt{false}_\mathtt{C}]$
$0.80 :: [\mathtt{true}_\mathtt{R}\ :-\ \mathtt{true}_\mathtt{C}]$	$0.20 :: [\mathtt{false}_\mathtt{R}\ :-\ \mathtt{true}_\mathtt{C}]$
$0.20 :: [\mathtt{true}_\mathtt{R}\ :-\ \mathtt{false}_\mathtt{C}]$	$0.80 :: [\mathtt{false}_\mathtt{R}\ :-\ \mathtt{false}_\mathtt{C}]$
$\mathbf{0.99} :: [\mathtt{\mathbf{true}}_\mathtt{W}\ :-\ \mathtt{\mathbf{true}}_\mathtt{R}, \mathtt{\mathbf{true}}_\mathtt{S}]$	$0.01 :: [\mathtt{false}_\mathtt{W}\ :-\ \mathtt{true}_\mathtt{R}, \mathtt{true}_\mathtt{S}]$
$0.90 :: [\mathtt{true}_\mathtt{W}\ :-\ \mathtt{true}_\mathtt{R}, \mathtt{false}_\mathtt{S}]$	$0.10 :: [\mathtt{false}_\mathtt{W}\ :-\ \mathtt{true}_\mathtt{R}, \mathtt{false}_\mathtt{S}]$
$0.90 :: [\mathtt{true}_\mathtt{W}\ :-\ \mathtt{false}_\mathtt{R}, \mathtt{true}_\mathtt{S}]$	$0.10 :: [\mathtt{false}_\mathtt{W}\ :-\ \mathtt{flase}_\mathtt{R}, \mathtt{true}_\mathtt{S}]$
$0.00 :: [\mathtt{true}_\mathtt{W}\ :-\ \mathtt{false}_\mathtt{R}, \mathtt{false}_\mathtt{S}]$	$1.00 :: [\mathtt{false}_\mathtt{W}\ :-\ \mathtt{flase}_\mathtt{R}, \mathtt{false}_\mathtt{S}]$

$$\cfrac{\cfrac{\cfrac{\cfrac{}{0.50 :: \mathtt{true}_\mathtt{C}, \mathtt{true}_\mathtt{C}}\ 0.50 :: [\mathtt{true}_\mathtt{C}, \mathtt{true}_\mathtt{C} : -.]}{0.40 = (0.50 \times 0.80) :: \mathtt{true}_\mathtt{C}, \mathtt{true}_\mathtt{R}}\ 0.80 :: [\mathtt{true}_\mathtt{R} : -\mathtt{true}_\mathtt{C}]}{0.04 = (0.40 \times 0.10) :: \mathtt{true}_\mathtt{R}, \mathtt{true}_\mathtt{S}}\ 0.10 :: [\mathtt{true}_\mathtt{S} : -\mathtt{true}_\mathtt{C}]}{\vdash 0.0396 = (0.04 \times 0.99) :: \mathtt{true}_\mathtt{W}}\ 0.99 :: [\mathtt{true}_\mathtt{W} : -\mathtt{true}_\mathtt{R}, \mathtt{true}_\mathtt{S}]$$

a single linear computation for checking a DAG and computing joint prob.

probLO inference: $$\mathrm{exp}\ \frac{q :: \ \mathrm{P}\ ;\ \Gamma, b_1, \ldots, b_m}{q \cdot p :: \ \mathrm{P}, \mathtt{M}\ ;\ \Gamma, h_1, \ldots, h_n}\, p :: [\mathtt{h}_1, \ldots, \mathtt{h}_\mathtt{n}] :- [\mathtt{b}_1, \ldots, \mathtt{b}_\mathtt{m}] = \mathtt{M}$$

probLO: methods and programs

- A **(Boolean) probLO-method** of **type** $X_1 \times \cdots \times X_n :- Y_1, \ldots, Y_n$ is a LO-method annotated with a probability value $p \in [0,1]$:

$$\mathtt{M} = p :: [\overbrace{\mathtt{w}_{\mathtt{X}_1}, \ldots, \mathtt{w}_{\mathtt{X}_n}}^{\text{multi-head}}] :- [\overbrace{\mathtt{w}_{\mathtt{Y}_1}, \ldots, \mathtt{w}_{\mathtt{Y}_m}}^{\text{body}}] \quad \text{or} \quad \mathtt{M} = p :: [\overbrace{\mathtt{w}_{\mathtt{X}_1}, \ldots, \mathtt{w}_{\mathtt{X}_n}}^{\text{multi-head}}] :- [\overbrace{\quad . \quad}^{\text{empty body}}] \;\; \text{if } m = 0$$

$\mathtt{w}_{\mathtt{X}_\mathtt{i}}$ is an atom/event of $\{\mathtt{tt}_{\mathtt{X}_\mathtt{i}}, \mathtt{ff}_{\mathtt{X}_\mathtt{i}}\}$ and $\mathtt{w}_{\mathtt{Y}_\mathtt{j}}$ an atom/event of $\{\mathtt{tt}_{\mathtt{Y}_\mathtt{j}}, \mathtt{ff}_{\mathtt{Y}_\mathtt{j}}\}$

$$\underbrace{\Pr(X_1, \ldots, X_n \mid Y_1, \ldots, Y_m) = p}_{\text{conditional probability}} \quad \rightsquigarrow \quad \underbrace{p}_{\text{probability}} :: [\underbrace{\mathtt{w}_{\mathtt{X}_1}, \ldots, \mathtt{w}_{\mathtt{X}_n}}_{\text{multi-head}}] :- [\underbrace{\mathtt{w}_{\mathtt{Y}_1}, \ldots, \mathtt{w}_{\mathtt{Y}_m}}_{\text{body}}]$$

- A **conditional distribution** of type $X_1 \times \cdots \times X_n :- Y_1, \ldots, Y_m$ given evidences $\mathtt{w}_{\mathtt{Y}_1}, \ldots, \mathtt{w}_{\mathtt{Y}_m}$ is the set of methods with same body $[\mathtt{w}_{\mathtt{Y}_1}, \ldots, \mathtt{w}_{\mathtt{Y}_m}]$

$$\mathtt{C}^{\mathtt{w}_{\mathtt{Y}_1}, \ldots, \mathtt{w}_{\mathtt{Y}_m}}_{X_1, \ldots, X_n} = \begin{bmatrix} p_1 :: [\mathtt{w}_{\mathtt{X}_1}{}^{\mathtt{1}}, \ldots, \mathtt{w}_{\mathtt{X}_n}{}^{\mathtt{1}}] :- [\mathtt{w}_{\mathtt{Y}_1}, \ldots, \mathtt{w}_{\mathtt{Y}_m}] \\ \vdots \\ p_i :: [\mathtt{w}_{\mathtt{X}_1}{}^{\mathtt{i}}, \ldots, \mathtt{w}_{\mathtt{X}_n}{}^{\mathtt{i}}] :- [\mathtt{w}_{\mathtt{Y}_1}, \ldots, \mathtt{w}_{\mathtt{Y}_m}] \\ \vdots \\ p_k :: [\mathtt{w}_{\mathtt{X}_1}{}^{\mathtt{k}}, \ldots, \mathtt{w}_{\mathtt{X}_n}{}^{\mathtt{k}}] :- [\mathtt{w}_{\mathtt{Y}_1}, \ldots, \mathtt{w}_{\mathtt{Y}_m}] \end{bmatrix} \text{ where } \textstyle\sum_{i=1}^{k} p_i = 1 \text{ , with } k = 2^n$$

- A **probLO program** is a set of probLO conditional distributions.

probLO: operational semantics

- **State**:

$$\text{probLO state:} \quad \vdash \underbrace{p}_{\textit{probability}} :: \underbrace{\mathsf{P}}_{\textit{program}} \; ; \; \underbrace{h_1, \dots, h_n, \Gamma}_{\textit{multi goal}}$$

- **Rules**: **expansion method** $\mathtt{M} \in \mathtt{T}$ (cond. distr.) or **branching**

$$\textit{termination}\; \frac{}{p :: \; \mathtt{T} \; ; \; \underbrace{h_1, \dots, h_n}_{\textit{trigger}}} \; p :: [\mathtt{h}_1, \dots, \mathtt{h}_\mathtt{n}] \mathrel{:-} [.] \in \mathtt{T}$$

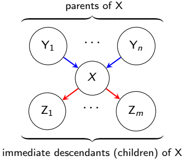
$$\textit{expansion + mix}\; \frac{q :: \; \mathsf{P} \; ; \; \Gamma}{q \cdot p :: \; \mathsf{P}, \mathtt{T} \; ; \; \Gamma, \underbrace{h_1, \dots, h_n}_{\textit{trigger}}} \; p :: [\mathtt{h}_1, \dots, \mathtt{h}_\mathtt{n}] \mathrel{:-} [.] \in \mathtt{T}$$

$$\textit{expansion}\; \frac{q :: \; \mathsf{P} \; ; \; \Gamma, b_1, \dots, b_m}{q \cdot p :: \; \mathsf{P}, \mathtt{T} \; ; \; \Gamma, \underbrace{h_1, \dots, h_n}_{\textit{trigger}}} \; p :: [\mathtt{h}_1, \dots, \mathtt{h}_\mathtt{n}] \mathrel{:-} [\mathtt{b}_1, \dots, \mathtt{b}_\mathtt{m}] \in \mathtt{T}$$

$$\textit{branching}\; \frac{p :: \; \mathsf{P} \; ; \; \Gamma, A \qquad\qquad q :: \; \mathsf{P} \; ; \; \Gamma, B}{p + q :: \; \mathsf{P} \; ; \; \Gamma, A \mathbin{\&} B}$$

Encoding (Boolean) Bayesian Networks in probLO

- $\mathcal{B} = \langle V, \rightharpoondown, \mathcal{P} \rangle$ is a Boolean Bayesian network.
- $\mathsf{X} \in V$ is a variable with parents $\{\mathsf{Y}_1, \dots, \mathsf{Y}_h\}$ and m children $\{\mathsf{Z}_1, \dots, \mathsf{Z}_m\}$
- CPT_{X} is the table associated to X
- every row r is mapped into a pair of probLO methods (a cond. prob. distr.)



a row of CPT_{X}:

⋮	⋮
$p_1 = \Pr(\mathsf{X} = \mathsf{True} \mid y_1, \dots, y_n)$	$p_2 = \Pr(\mathsf{X} = \mathsf{False} \mid y_1, \dots, y_n)$
⋮	⋮

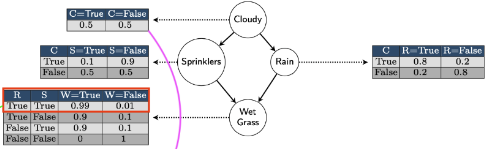
$$\mathtt{C}_{\mathsf{X}}^{\mathtt{w}_{\mathsf{Y}_1},\dots,\mathtt{w}_{\mathsf{Y}_n}} = \left[\begin{array}{l} p_1 :: [\underbrace{\mathtt{tt}_{\mathsf{X}}, \dots, \mathtt{tt}_{\mathsf{X}}}_{\mathtt{m}\ \text{times}}] :- [\mathtt{w}_{\mathsf{Y}_1}, \dots, \mathtt{w}_{\mathsf{Y}_n}] \\ \\ p_2 :: [\underbrace{\mathtt{ff}_{\mathsf{X}}, \dots, \mathtt{ff}_{\mathsf{X}}}_{\mathtt{m}\ \text{times}}] :- [\mathtt{w}_{\mathsf{Y}_1}, \dots, \mathtt{w}_{\mathsf{Y}_n}] \end{array}\right]$$ with $p_1 + p_2 = 1$ and heads with the same length m

where $\mathtt{w}_{\mathsf{Y}_\mathtt{i}}$ denotes $\mathtt{tt}_{\mathsf{Y}_\mathtt{i}}$ (if y_i is the event $\mathsf{Y}_i = \mathit{True}$) or $\mathtt{ff}_{\mathsf{Y}_\mathtt{i}}$, otherwise

- the **probLO table** of X: $\mathtt{T}_{\mathsf{X}} = \bigcup_{j=1}^{n+1} \mathtt{C}_{\mathsf{X}}^{\mathtt{w}_{\mathsf{Y}_1}{}^{\mathtt{j}},\dots,\mathtt{w}_{\mathsf{Y}_n}{}^{\mathtt{j}}}$
- the **probLO program**: $\mathtt{P}_{\mathcal{B}} = \bigcup_{\mathsf{X} \in V} \mathtt{T}_{\mathsf{X}}$

Example: encoding a BN in probLO

the probLO Program (multiset of tables) $\left[T_C, T_W, T_S, T_R\right]$ associated to BN



C=True	C=False
0.5	0.5

C	S=True	S=False
True	0.1	0.9
False	0.5	0.5

C	R=True	R=False
True	0.8	0.2
False	0.2	0.8

R	S	W=True	W=False
True	True	0.99	0.01
True	False	0.9	0.1
False	True	0.9	0.1
False	False	0	1

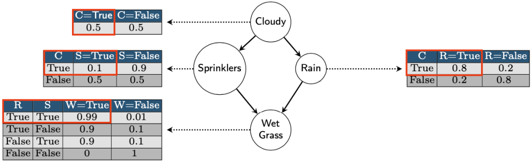
Conditional probability	**probLO Method**	**Conditional distribution**	**Table**
Pr(C = True) = 0.5 Pr(C = False) = 0.5	$M_C^{\langle 1\rangle} = 0.5 :: [\mathtt{tt}_C, \mathtt{tt}_C, \mathtt{tt}_C] :- [.]$ $M_C^{\langle 2\rangle} = 0.5 :: [\mathtt{ff}_C, \mathtt{ff}_C, \mathtt{ff}_C] :- [.]$	C_C	T_C
Pr(R = True | C = False) = 0.2 Pr(R = False | C = False) = 0.8	$M_R^{\langle 1\rangle} = 0.2 :: [\mathtt{tt}_R, \mathtt{tt}_R] :- [\mathtt{ff}_C]$ $M_R^{\langle 2\rangle} = 0.8 :: [\mathtt{ff}_R, \mathtt{ff}_R] :- [\mathtt{ff}_C]$	$C_R^{\mathtt{ff}_C}$	T_R
Pr(R = True | C = True) = 0.8 Pr(R = False | C = True) = 0.2	$M_R^{\langle 3\rangle} = 0.8 :: [\mathtt{tt}_R, \mathtt{tt}_R] :- [\mathtt{tt}_C]$ $M_R^{\langle 4\rangle} = 0.2 :: [\mathtt{ff}_R, \mathtt{ff}_R] :- [\mathtt{tt}_C]$	$C_R^{\mathtt{tt}_C}$	
Pr(S = True | C = False) = 0.5 Pr(S = False | C = False) = 0.5	$M_S^{\langle 1\rangle} = 0.5 :: [\mathtt{tt}_S, \mathtt{tt}_S] :- [\mathtt{ff}_C]$ $M_S^{\langle 2\rangle} = 0.5 :: [\mathtt{ff}_S, \mathtt{ff}_S] :- [\mathtt{ff}_C]$	$C_S^{\mathtt{ff}_C}$	T_S
Pr(S = True | C = True) = 0.1 Pr(S = False | C = True) = 0.9	$M_S^{\langle 3\rangle} = 0.1 :: [\mathtt{tt}_S, \mathtt{tt}_S] :- [\mathtt{tt}_C]$ $M_S^{\langle 4\rangle} = 0.9 :: [\mathtt{ff}_S, \mathtt{ff}_S] :- [\mathtt{tt}_C]$	$C_S^{\mathtt{tt}_C}$	
Pr(W = True | R = False, S = False) = 0 Pr(W = False | R = False, S = False) = 1	$M_W^{\langle 1\rangle} = 0 :: [\mathtt{tt}_W] :- [\mathtt{ff}_S, \mathtt{ff}_R]$ $M_W^{\langle 2\rangle} = 1 :: [\mathtt{ff}_W] :- [\mathtt{ff}_S, \mathtt{ff}_R]$	$C_W^{\mathtt{ff}_R, \mathtt{ff}_S}$	T_W
Pr(W = True | R = False, S = True) = 0.9 Pr(W = False | R = False, S = True) = 0.1	$M_W^{\langle 3\rangle} = 0.9 :: [\mathtt{tt}_W] :- [\mathtt{tt}_S, \mathtt{ff}_R]$ $M_W^{\langle 4\rangle} = 0.1 :: [\mathtt{ff}_W] :- [\mathtt{tt}_S, \mathtt{ff}_R]$	$C_W^{\mathtt{ff}_R, \mathtt{tt}_S}$	
Pr(W = True | R = True, S = False) = 0.9 Pr(W = False | R = True, S = False) = 0.1	$M_W^{\langle 5\rangle} = 0.9 :: [\mathtt{tt}_W] :- [\mathtt{ff}_S, \mathtt{tt}_R]$ $M_W^{\langle 6\rangle} = 0.1 :: [\mathtt{ff}_W] :- [\mathtt{ff}_S, \mathtt{tt}_R]$	$C_W^{\mathtt{tt}_R, \mathtt{ff}_S}$	
Pr(W = True | R = True, S = True) = 0.99 Pr(W = False | R = True, S = True) = 0.01	$M_W^{\langle 7\rangle} = 0.99 :: [\mathtt{tt}_W] :- [\mathtt{tt}_S, \mathtt{tt}_R]$ $M_W^{\langle 8\rangle} = 0.01 :: [\mathtt{ff}_W] :- [\mathtt{tt}_S, \mathtt{tt}_R]$	$C_W^{\mathtt{tt}_R, \mathtt{ff}_S}$	

probLO correctly computes joint probabilities

Theorem (Computing joint probability with probLO)

If $\mathcal{B} = \langle V, \rightharpoonup, \mathcal{P} \rangle$ *is a (Boolean) BN with variables* $V = \{\mathrm{X}_1, \ldots, \mathrm{X}_n\}$ *then: state* $p :: \; \mathrm{P}_{\mathcal{B}} \; ; \; \mathbb{w}_{\mathrm{X}_1}, \ldots, \mathbb{w}_{\mathrm{X}_n}$ *is derivable in probLO (at the cost* $O(n)$*) iff* $p = \Pr(x_1 \ldots, x_n)$ *is the joint probability of all variables in* $\mathcal{B}$*.*

Bayesian Network:



C=True	C=False
0.5	0.5

C	S=True	S=False
True	0.1	0.9
False	0.5	0.5

C	R=True	R=False
True	0.8	0.2
False	0.2	0.8

R	S	W=True	W=False
True	True	0.99	0.01
True	False	0.9	0.1
False	True	0.9	0.1
False	False	0	1

$$\Pr(\mathtt{C} = \mathrm{True}, \mathtt{R} = \mathrm{True}, \mathtt{S} = \mathrm{True}, \mathtt{W} = \mathrm{True}) = \underbrace{\Pr(\mathrm{W} = \mathrm{True} \mid \mathrm{R} = \mathrm{True}, \mathrm{S} = \mathrm{True})}_{0.99::\mathtt{M}_{\mathrm{W}}^{7}} \times \underbrace{\Pr(\mathrm{S} = \mathrm{True} \mid \mathrm{C} = \mathrm{True})}_{0.1::\mathtt{M}_{\mathrm{S}}^{3}} \times \underbrace{\Pr(\mathrm{R} = \mathrm{True} \mid \mathrm{C} = \mathrm{True})}_{0.8::\mathtt{M}_{\mathrm{R}}^{3}} \times \underbrace{\Pr(\mathrm{C} = \mathrm{True})}_{0.5::\mathtt{M}_{\mathrm{C}}^{1}}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{{\color{red}0.5} :: \; \mathtt{T}_{\mathrm{C}} \; ; \; {\color{red}\mathtt{tt}_{\mathrm{C}}, \mathtt{tt}_{\mathrm{C}}, \mathtt{tt}_{\mathrm{C}}}} \; \mathtt{M}_{\mathrm{C}}^{\langle 1 \rangle} = 0.5 :: [\mathtt{tt}_{\mathrm{C}}, \mathtt{tt}_{\mathrm{C}}, \mathtt{tt}_{\mathrm{C}}] :- []}{{\color{red}0.8} \times 0.5 :: \; \mathtt{T}_{\mathrm{C}}, \mathtt{T}_{\mathrm{R}} \; ; \; \mathtt{tt}_{\mathrm{C}}, \mathtt{tt}_{\mathrm{C}}, {\color{red}\mathtt{tt}_{\mathrm{R}}, \mathtt{tt}_{\mathrm{R}}}} \; \mathtt{M}_{\mathrm{R}}^{\langle 3 \rangle} = 0.8 :: [\mathtt{tt}_{\mathrm{R}}, \mathtt{tt}_{\mathrm{R}}] :- [\mathtt{tt}_{\mathrm{C}}]}{{\color{red}0.1} \times 0.4 :: \; \mathtt{T}_{\mathrm{C}}, \mathtt{T}_{\mathrm{R}}, \mathtt{T}_{\mathrm{S}} \; ; \; \mathtt{tt}_{\mathrm{C}}, \mathtt{tt}_{\mathrm{R}}, \mathtt{tt}_{\mathrm{R}}, {\color{red}\mathtt{tt}_{\mathrm{S}}, \mathtt{tt}_{\mathrm{S}}}} \; \mathtt{M}_{\mathrm{S}}^{\langle 3 \rangle} = 0.1 :: [\mathtt{tt}_{\mathrm{S}}, \mathtt{tt}_{\mathrm{S}}] :- [\mathtt{tt}_{\mathrm{C}}]}{\underbrace{{\color{red}0.99} \times 0.04}_{0.0396} :: \; \mathtt{T}_{\mathrm{C}}, \mathtt{T}_{\mathrm{R}}, \mathtt{T}_{\mathrm{S}}, \mathtt{T}_{\mathrm{W}} \; ; \; \mathtt{tt}_{\mathrm{C}}, \mathtt{tt}_{\mathrm{R}}, {\color{red}\mathtt{tt}_{\mathrm{W}}}, \mathtt{tt}_{\mathrm{S}}} \; \mathtt{M}_{\mathrm{W}}^{\langle 7 \rangle} = 0.99 :: [\mathtt{tt}_{\mathrm{W}}] :- [\mathtt{tt}_{\mathrm{S}}, \mathtt{tt}_{\mathrm{R}}]$$

probLO correctly computes marginal probabilities

Theorem (Computing marginal probability with probLO)

If $\mathcal{B} = \langle V, \rightharpoonup, \mathcal{P}\rangle$ *is a (Boolean) BN with variables* $V = \{\mathrm{X}_1, \ldots, \mathrm{X}_n\}$ *then, the state* $p :: \mathrm{P}_{\mathcal{B}} \,;\, \mathtt{w}_{\mathrm{X}_1}, \ldots, \mathtt{w}_{\mathrm{X}_m}, (\mathtt{tt}_{\mathrm{X}_{m+1}} \,\&\, \mathtt{ff}_{\mathrm{X}_{m+1}}), \ldots, (\mathtt{tt}_{\mathrm{X}_n} \,\&\, \mathtt{ff}_{\mathrm{X}_n})$ *is derivable in probLO (at the cost* $\mathcal{O}(2^k n)$*) iff* $p = \Pr(X_1 \ldots, X_m)$ *is the marginal probability of* $\mathrm{X}_1, \ldots, \mathrm{X}_m$ *regardless of variables* $\mathrm{X}_{m+1}, \ldots, \mathrm{X}_n$

$$\underbrace{\Pr(\mathrm{C} = \mathrm{True}, \mathrm{R} = \mathrm{True}, \mathrm{W} = \mathrm{False})}_{\text{marginal probability w.r.t. S}} = \left\{ \begin{array}{c} \overbrace{\Pr(\mathrm{W} = \mathrm{False} \mid \mathrm{R} = \mathrm{True}, \mathrm{S} = \mathrm{False})}^{0.1::\mathtt{M}^6_{\mathrm{W}}} \times \overbrace{\Pr(\mathrm{S} = \mathrm{False} \mid \mathrm{C} = \mathrm{True})}^{0.9::\mathtt{M}^4_{\mathrm{S}}} \times \overbrace{\Pr(\mathrm{R} = \mathrm{True} \mid \mathrm{C} = \mathrm{True})}^{0.8::\mathtt{M}^3_{\mathrm{R}}} \times \overbrace{\Pr(\mathrm{C} = \mathrm{True})}^{0.5::\mathtt{M}^1_{\mathrm{C}}} \\ + \\ \underbrace{\Pr(\mathrm{W} = \mathrm{False} \mid \mathrm{R} = \mathrm{True}, \mathrm{S} = \mathrm{True})}_{0.01::\mathtt{M}^8_{\mathrm{W}}} \times \underbrace{\Pr(\mathrm{S} = \mathrm{True} \mid \mathrm{C} = \mathrm{True})}_{0.1::\mathtt{M}^3_{\mathrm{S}}} \times \underbrace{\Pr(\mathrm{R} = \mathrm{True} \mid \mathrm{C} = \mathrm{True})}_{0.8::\mathtt{M}^3_{\mathrm{R}}} \times \underbrace{\Pr(\mathrm{C} = \mathrm{True})}_{0.5::\mathtt{M}^1_{\mathrm{C}}} \end{array} \right\}$$

$$\text{bra}\;\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{0.5 :: \mathtt{T}_{\mathrm{C}} \,;\, \mathtt{tt}_{\mathrm{C}}, \mathtt{tt}_{\mathrm{C}}, \mathtt{tt}_{\mathrm{C}}}\;\mathtt{M}_{\mathrm{C}}^{\langle 1\rangle} = 0.5 :: [\mathtt{tt}_{\mathrm{C}}, \mathtt{tt}_{\mathrm{C}}, \mathtt{tt}_{\mathrm{C}}] :- []}{0.8 \times 0.5 :: \mathtt{T}_{\mathrm{C}}, \mathtt{T}_{\mathrm{R}} \,;\, \mathtt{tt}_{\mathrm{C}}, \mathtt{tt}_{\mathrm{R}}, \mathtt{tt}_{\mathrm{R}}, \mathtt{tt}_{\mathrm{C}}}\;\mathtt{M}_{\mathrm{R}}^{\langle 3\rangle} = 0.8 :: [\mathtt{tt}_{\mathrm{R}}, \mathtt{tt}_{\mathrm{R}}] :- [\mathtt{tt}_{\mathrm{C}}]}{0.9 \times 0.4 :: \mathtt{T}_{\mathrm{C}}, \mathtt{T}_{\mathrm{R}}, \mathtt{T}_{\mathrm{S}} \,;\, \mathtt{tt}_{\mathrm{C}}, \mathtt{tt}_{\mathrm{R}}, \mathtt{tt}_{\mathrm{R}}, \mathtt{ff}_{\mathrm{S}}, \mathtt{ff}_{\mathrm{S}}}\;\mathtt{M}_{\mathrm{S}}^{\langle 4\rangle} = 0.9 :: [\mathtt{ff}_{\mathrm{S}}, \mathtt{ff}_{\mathrm{S}}] :- [\mathtt{tt}_{\mathrm{C}}]}{0.1 \times 0.36 :: \mathtt{T}_{\mathrm{C}}, \mathtt{T}_{\mathrm{R}}, \mathtt{T}_{\mathrm{S}}, \mathtt{T}_{\mathrm{W}} \,;\, \mathtt{tt}_{\mathrm{C}}, \mathtt{tt}_{\mathrm{R}}, \mathtt{ff}_{\mathrm{W}}, \mathtt{ff}_{\mathrm{S}}}\;\mathtt{M}_{\mathrm{W}}^{\langle 6\rangle} = 0.1 :: [\mathtt{ff}_{\mathrm{W}}] :- [\mathtt{ff}_{\mathrm{S}}, \mathtt{tt}_{\mathrm{R}}] \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{0.5 :: \mathtt{T}_{\mathrm{C}} \,;\, \mathtt{tt}_{\mathrm{C}}, \mathtt{tt}_{\mathrm{C}}, \mathtt{tt}_{\mathrm{C}}}\;\mathtt{M}_{\mathrm{C}}^{\langle 1\rangle} = 0.5 :: [\mathtt{tt}_{\mathrm{C}}, \mathtt{tt}_{\mathrm{C}}, \mathtt{tt}_{\mathrm{C}}] :- []}{0.8 \times 0.5 :: \mathtt{T}_{\mathrm{C}}, \mathtt{T}_{\mathrm{R}} \,;\, \mathtt{tt}_{\mathrm{C}}, \mathtt{tt}_{\mathrm{R}}, \mathtt{tt}_{\mathrm{R}}, \mathtt{tt}_{\mathrm{C}}}\;\mathtt{M}_{\mathrm{R}}^{\langle 3\rangle} = 0.8 :: [\mathtt{tt}_{\mathrm{R}}, \mathtt{tt}_{\mathrm{R}}] :- [\mathtt{tt}_{\mathrm{C}}]}{0.1 \times 0.4 :: \mathtt{T}_{\mathrm{C}}, \mathtt{T}_{\mathrm{R}}, \mathtt{T}_{\mathrm{S}} \,;\, \mathtt{tt}_{\mathrm{C}}, \mathtt{tt}_{\mathrm{R}}, \mathtt{tt}_{\mathrm{R}}, \mathtt{tt}_{\mathrm{S}}, \mathtt{tt}_{\mathrm{S}}}\;\mathtt{M}_{\mathrm{S}}^{\langle 3\rangle} = 0.1 :: [\mathtt{tt}_{\mathrm{S}}, \mathtt{tt}_{\mathrm{S}}] :- [\mathtt{tt}_{\mathrm{C}}]}{0.01 \times 0.04 :: \mathtt{T}_{\mathrm{C}}, \mathtt{T}_{\mathrm{R}}, \mathtt{T}_{\mathrm{S}}, \mathtt{T}_{\mathrm{W}} \,;\, \mathtt{tt}_{\mathrm{C}}, \mathtt{tt}_{\mathrm{R}}, \mathtt{ff}_{\mathrm{W}}, \mathtt{tt}_{\mathrm{S}}}\;\mathtt{M}_{\mathrm{W}}^{\langle 8\rangle} = 0.01 :: [\mathtt{ff}_{\mathrm{W}}] :- [\mathtt{tt}_{\mathrm{S}}, \mathtt{tt}_{\mathrm{R}}]}{\underbrace{0.036 + 0.0004}_{0.0364} :: \mathtt{T}_{\mathrm{C}}, \mathtt{T}_{\mathrm{R}}, \mathtt{T}_{\mathrm{S}}, \mathtt{T}_{\mathrm{W}} \,;\, \mathtt{tt}_{\mathrm{R}}, \mathtt{tt}_{\mathrm{C}}, \mathtt{tt}_{\mathrm{R}}, \mathtt{ff}_{\mathrm{W}}, \mathtt{ff}_{\mathrm{S}} \,\&\, \mathtt{tt}_{\mathrm{S}}}$$

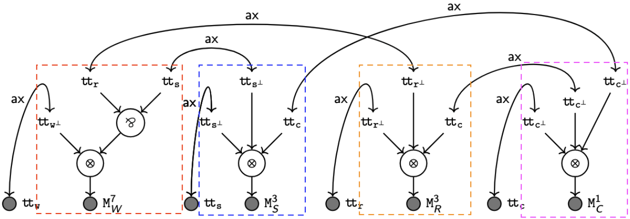
Conclusions: probLO and proof nets of linear logic

probLO as a graphical language!

any probLO derivation computing a joint probability (a tree)

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{0.5 :: \mathtt{T}_C \;;\; \mathtt{tt}_C, \mathtt{tt}_C, \mathtt{tt}_C}\ \mathtt{M}_C^{\langle 1\rangle} = 0.5 :: [\mathtt{tt}_C, \mathtt{tt}_C, \mathtt{tt}_C] \coloneq []}{0.8 \times 0.5 :: \mathtt{T}_C, \mathtt{T}_R \;;\; \mathtt{tt}_C, \mathtt{tt}_C, \mathtt{tt}_R, \mathtt{tt}_R}\ \mathtt{M}_R^{\langle 3\rangle} = 0.8 :: [\mathtt{tt}_R, \mathtt{tt}_R] \coloneq [\mathtt{tt}_C]}{0.1 \times 0.4 :: \mathtt{T}_C, \mathtt{T}_R, \mathtt{T}_S \;;\; \mathtt{tt}_C, \mathtt{tt}_R, \mathtt{tt}_R, \mathtt{tt}_S, \mathtt{tt}_S}\ \mathtt{M}_S^{\langle 3\rangle} = 0.1 :: [\mathtt{tt}_S, \mathtt{tt}_S] \coloneq [\mathtt{tt}_C]}{\underbrace{0.99 \times 0.04}_{0.0396} :: \mathtt{T}_C, \mathtt{T}_R, \mathtt{T}_S, \mathtt{T}_W \;;\; \mathtt{tt}_C, \mathtt{tt}_R, \mathtt{tt}_W, \mathtt{tt}_S}\ \mathtt{M}_W^{\langle 7\rangle} = 0.99 :: [\mathtt{tt}_W] \coloneq [\mathtt{tt}_S, \mathtt{tt}_R]
$$

can be transformed into a MLL proof net (a graph)



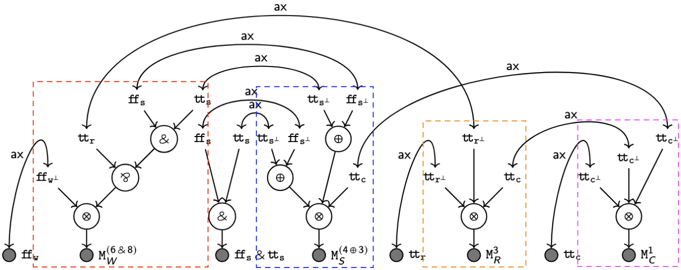
Conclusions: probLO and proof nets of linear logic

probLO as a graphical language!

any probLO derivation computing a marginal probability (a tree)

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{$\mathtt{M}^1_C$}
\UnaryInfC{$0.5 :: \mathtt{T}_C \,;\, \mathtt{tt}_C, \mathtt{tt}_C, \mathtt{tt}_C$}
\LeftLabel{$\mathtt{M}^3_R$}
\UnaryInfC{$0.8 \times 0.5 :: \mathtt{T}_C, \mathtt{T}_R \,;\, \mathtt{tt}_C, \mathtt{tt}_R, \mathtt{tt}_R, \mathtt{tt}_C$}
\LeftLabel{$\mathtt{M}^4_S$}
\UnaryInfC{$0.9 \times 0.4 :: \mathtt{T}_C, \mathtt{T}_R, \mathtt{T}_S \,;\, \mathtt{tt}_C, \mathtt{tt}_R, \mathtt{tt}_R, \mathtt{ff}_S, \mathtt{ff}_S$}
\LeftLabel{$\mathtt{M}^6_W$}
\UnaryInfC{$0.1 \times 0.36 :: \mathtt{T}_C, \mathtt{T}_R, \mathtt{T}_S, \mathtt{T}_W \,;\, \mathtt{tt}_C, \mathtt{tt}_R, \mathtt{ff}_W, \mathtt{ff}_S$}
\AxiomC{}
\LeftLabel{$\mathtt{M}^1_C$}
\UnaryInfC{$0.5 :: \mathtt{T}_C \,;\, \mathtt{tt}_C, \mathtt{tt}_C, \mathtt{tt}_C$}
\LeftLabel{$\mathtt{M}^3_R$}
\UnaryInfC{$0.8 \times 0.5 :: \mathtt{T}_C, \mathtt{T}_R \,;\, \mathtt{tt}_C, \mathtt{tt}_R, \mathtt{tt}_R, \mathtt{tt}_C$}
\LeftLabel{$\mathtt{M}^3_S$}
\UnaryInfC{$0.1 \times 0.4 :: \mathtt{T}_C, \mathtt{T}_R, \mathtt{T}_S \,;\, \mathtt{tt}_C, \mathtt{tt}_R, \mathtt{tt}_R, \mathtt{tt}_S, \mathtt{tt}_S$}
\LeftLabel{$\mathtt{M}^8_W$}
\UnaryInfC{$0.01 \times 0.04 :: \mathtt{T}_C, \mathtt{T}_R, \mathtt{T}_S, \mathtt{T}_W \,;\, \mathtt{tt}_C, \mathtt{tt}_R, \mathtt{ff}_W, \mathtt{tt}_S$}
\LeftLabel{bra}
\BinaryInfC{$\underbrace{0.036 + 0.0004}_{0.0364} :: \mathtt{T}_C, \mathtt{T}_R, \mathtt{T}_S, \mathtt{T}_W \,;\, \mathtt{tt}_R, \mathtt{tt}_C, \mathtt{tt}_R, \mathtt{ff}_W, \mathtt{ff}_S \,\&\, \mathtt{tt}_S$}
\end{prooftree}
$$

can be transformed into a MALL proof net (a graph)



Conclusions: Related works on PLP and Future work

- Most PLP languages extend classical LP by attaching probabilities to facts or to rule heads, enabling probabilistic inference through weighted model counting or explanation-based inference.
- In probLO, the probabilistic aspect is limited to the measure of the reachability of the goal, and the non-deterministic choices made during proof search are controlled by the goal itself (the "branching rule").
- A future work is to implement optimization techniques for Bayesian inference within probLO as, e.g.:
  - algorithms such as *clique tree/junction tree* [Lauritzen '88, Jensen, '96]
  - or *variable elimination* [Zhang, '94] and *belief propagation* [Pearl, '88].

Thank you!