A Fine-Grained Small-Step Semantics for Interleaving Search

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Brysen Pfingsten Jason Hemann

Seton Hall University

{**pfingsbr**, hemannja}@shu.edu

Interleaving DFS

Interleaving DFS

$$\texttt{DOGS}(x) \leftarrow (x = \texttt{dog}) \vee \texttt{DOGS}(x)$$

$$\texttt{CATS}(x) \leftarrow (x = \texttt{cat}) \vee \texttt{CATS}(x)$$

$$\texttt{DOGS-CATS}(x) \leftarrow \texttt{DOGS}(x) \vee \texttt{CATS}(x)$$

$$\texttt{DOGS-CATS}(q)$$

Interleaving DFS

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$$\texttt{DOGS-CATS}(q)$$

DFS (Prolog)

```
dog, dog, dog, dog, ...
```

Interleaving DFS

$$\texttt{DOGS}(x) \leftarrow (x = \texttt{dog}) \vee \texttt{DOGS}(x)$$

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$$\texttt{DOGS-CATS}(x) \leftarrow \texttt{DOGS}(x) \vee \texttt{CATS}(x)$$

$$\texttt{DOGS-CATS}(q)$$

DFS (Prolog)

```
dog, dog, dog, dog, ...
```

Interleaving DFS (miniKanren)

```
dog, cat, dog, cat, ...
```

Interleaving DFS

$$\texttt{SAME}(x, y) \leftarrow (x = y)$$

$$(\texttt{SAME}(q, \texttt{turtle}) \vee \texttt{SAME}(q, \texttt{cat}) \vee (q = \texttt{dog})) \vee \texttt{SAME}(q, \texttt{fish})$$

Interleaving DFS

$$\texttt{SAME}(x, y) \leftarrow (x = y)$$

$$(\texttt{SAME}(q, \texttt{turtle}) \vee \texttt{SAME}(q, \texttt{cat}) \vee (q = \texttt{dog})) \vee \texttt{SAME}(q, \texttt{fish})$$

DFS (Prolog)

```
turtle, cat, dog, fish
```

Interleaving DFS

$$\texttt{SAME}(x, y) \leftarrow (x = y)$$

$$(\texttt{SAME}(q, \texttt{turtle}) \vee \texttt{SAME}(q, \texttt{cat}) \vee (q = \texttt{dog})) \vee \texttt{SAME}(q, \texttt{fish})$$

DFS (Prolog)

```
turtle, cat, dog, fish
```

Interleaving DFS (miniKanren)

```
fish, turtle, dog, cat
```

Interleaving DFS

- Suspends and resumes evaluation of branches

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- Fair and complete search
  - Kiselyov et al. 2005, Rozplokhas et al. 2020
  - Avoids divergence on an infinite branch

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- Suspends and resumes evaluation of branches
- Fair and complete search
  - Kiselyov et al. 2005, Rozplokhas et al. 2020
  - Avoids divergence on an infinite branch
- **Unintuitive behavior**

Calculi, Expressions, Values

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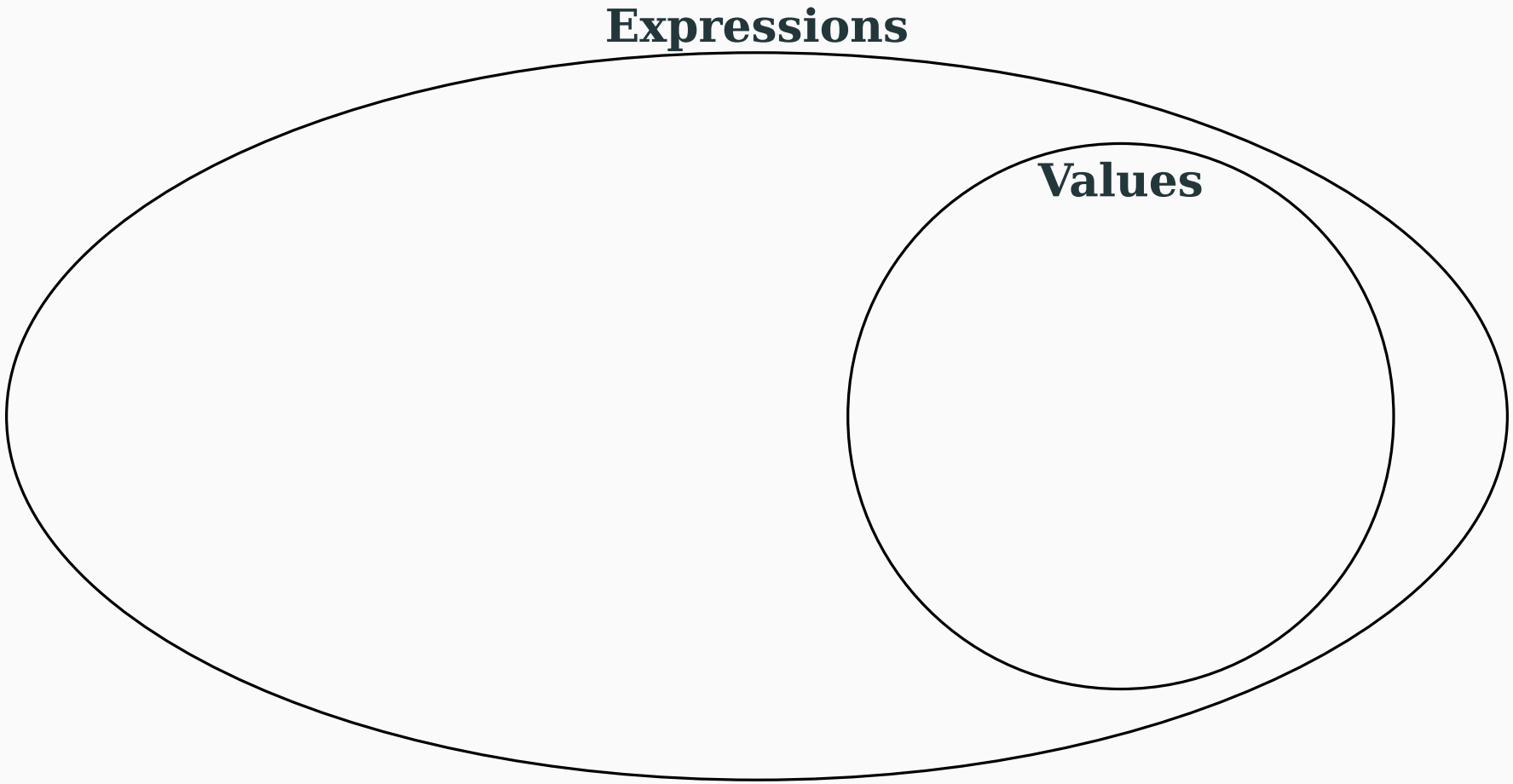

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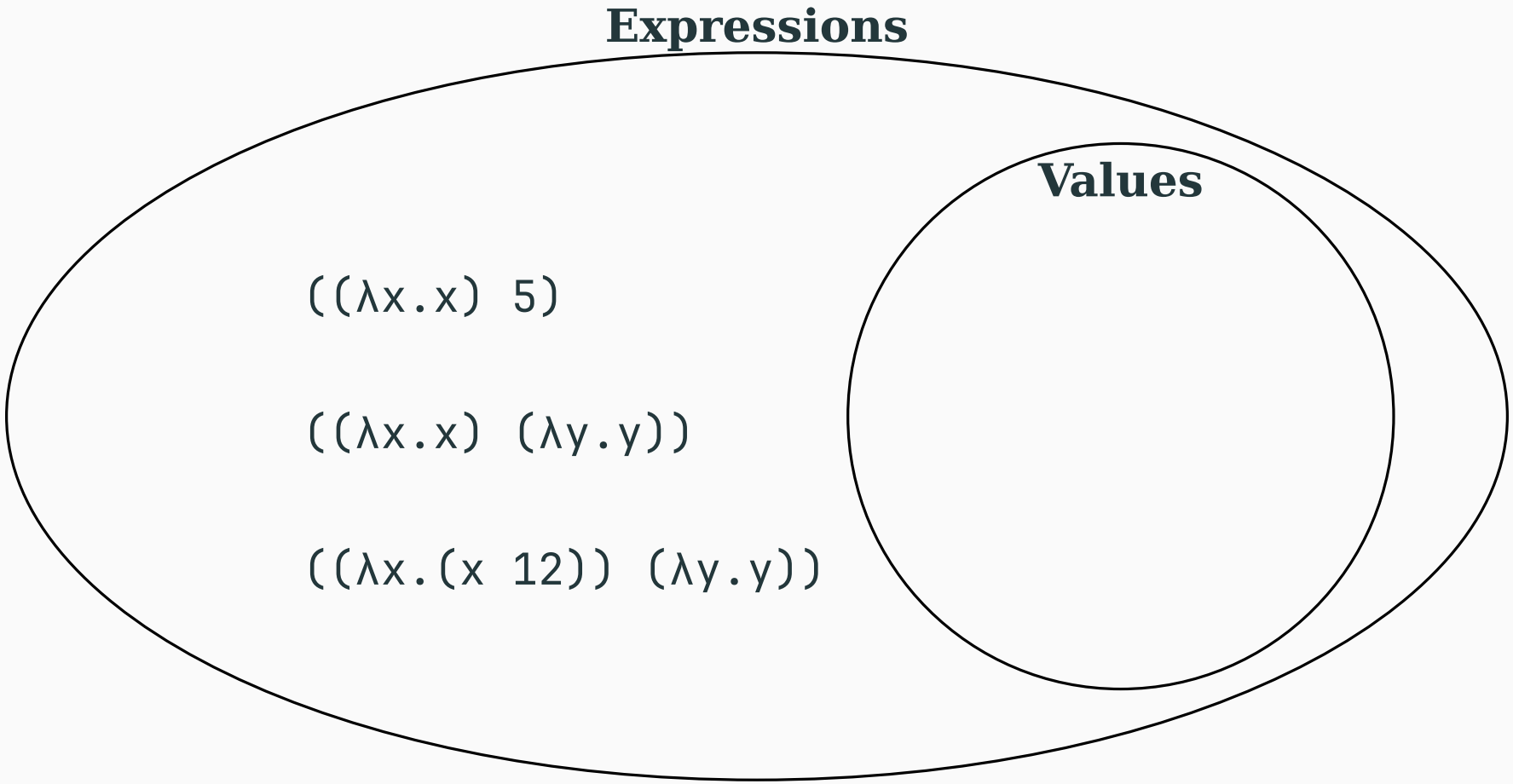

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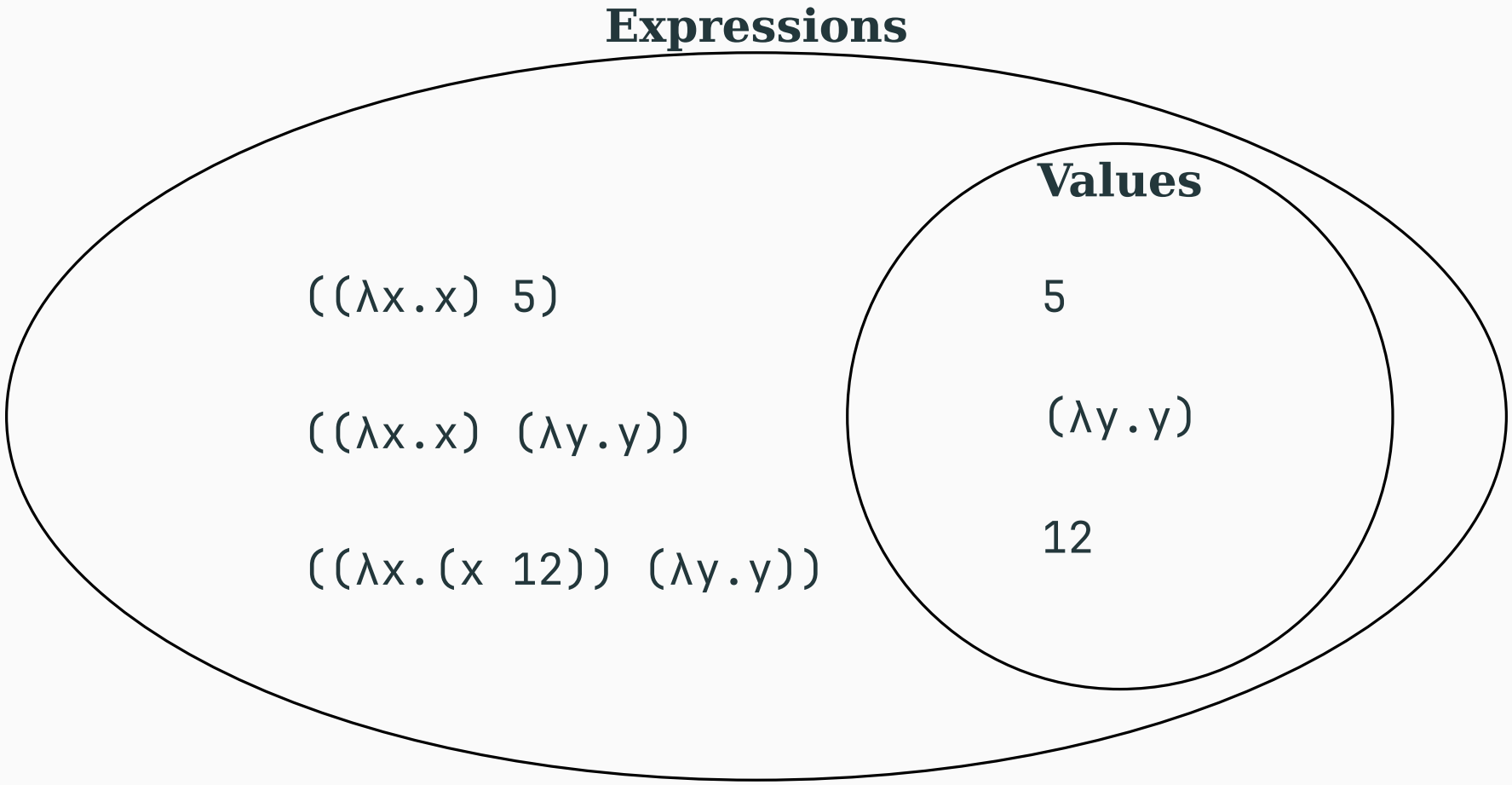

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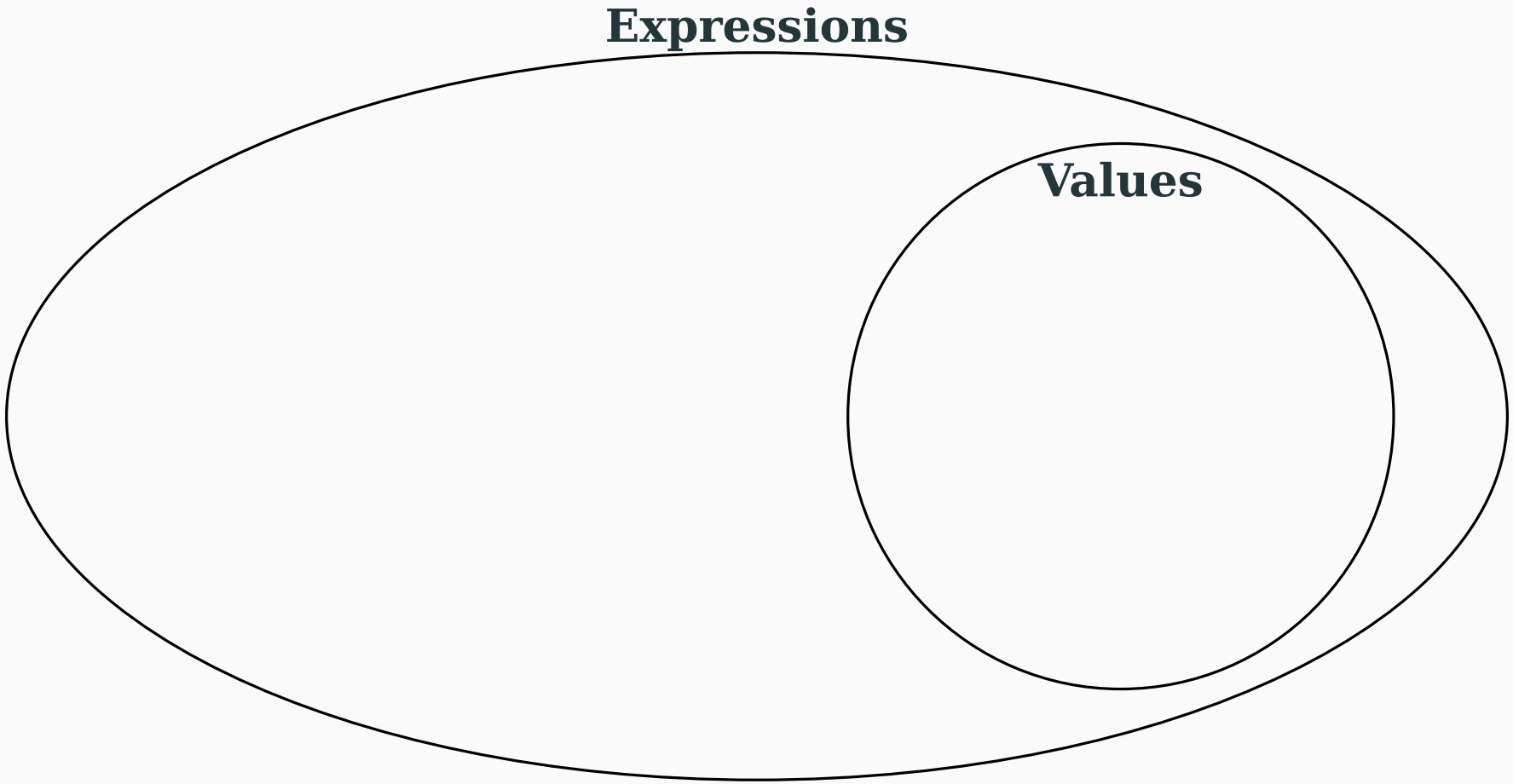

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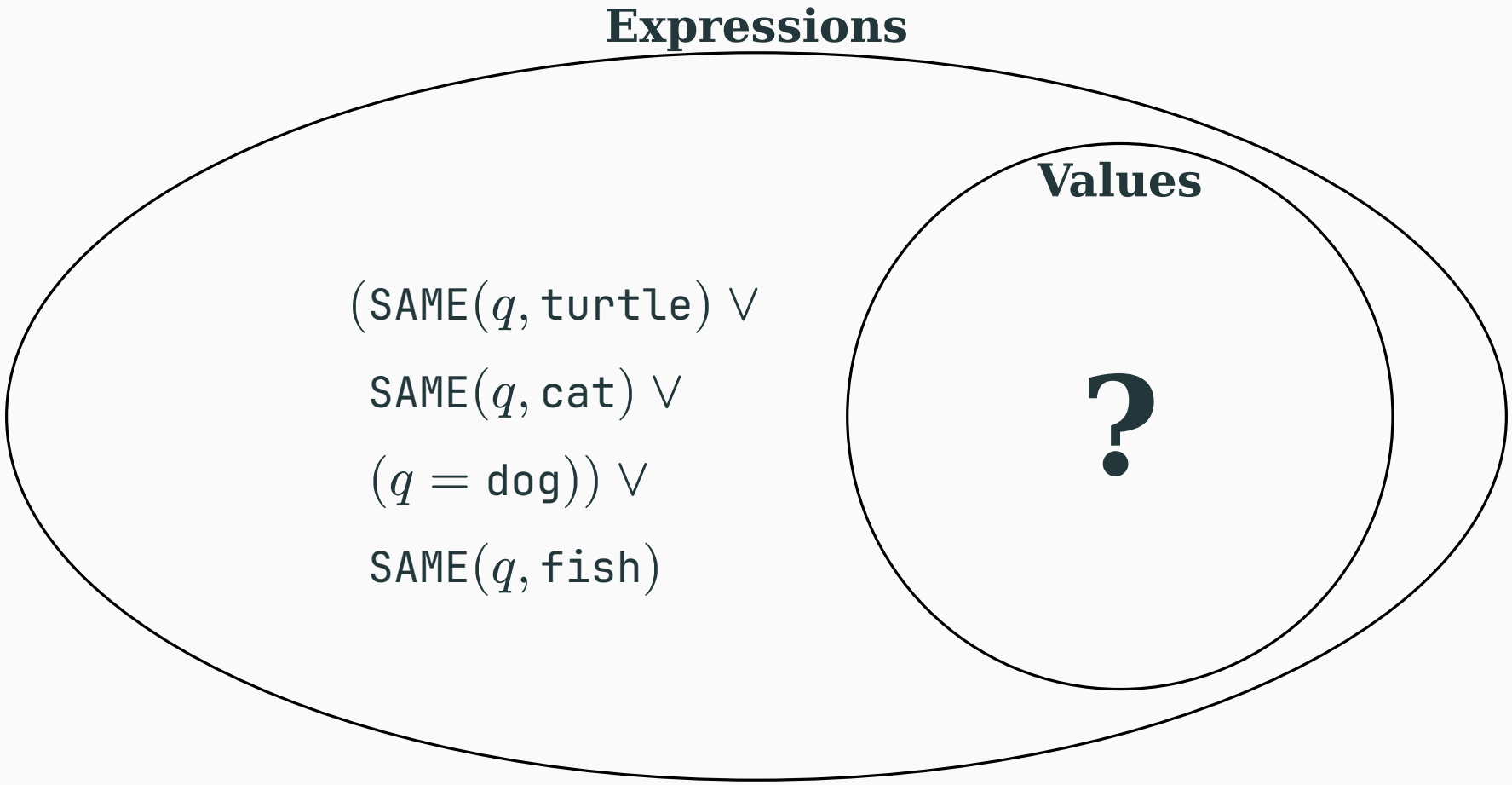

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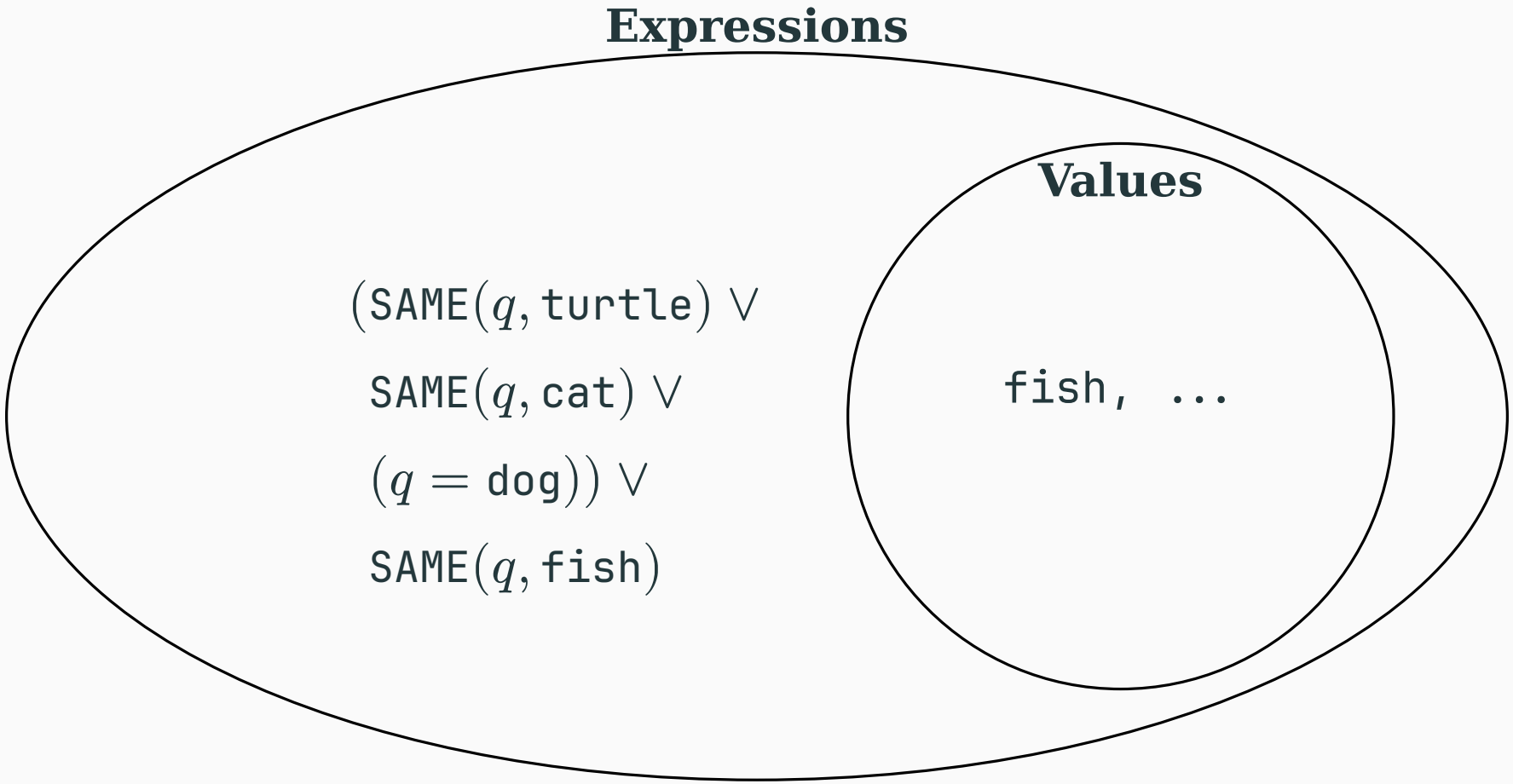

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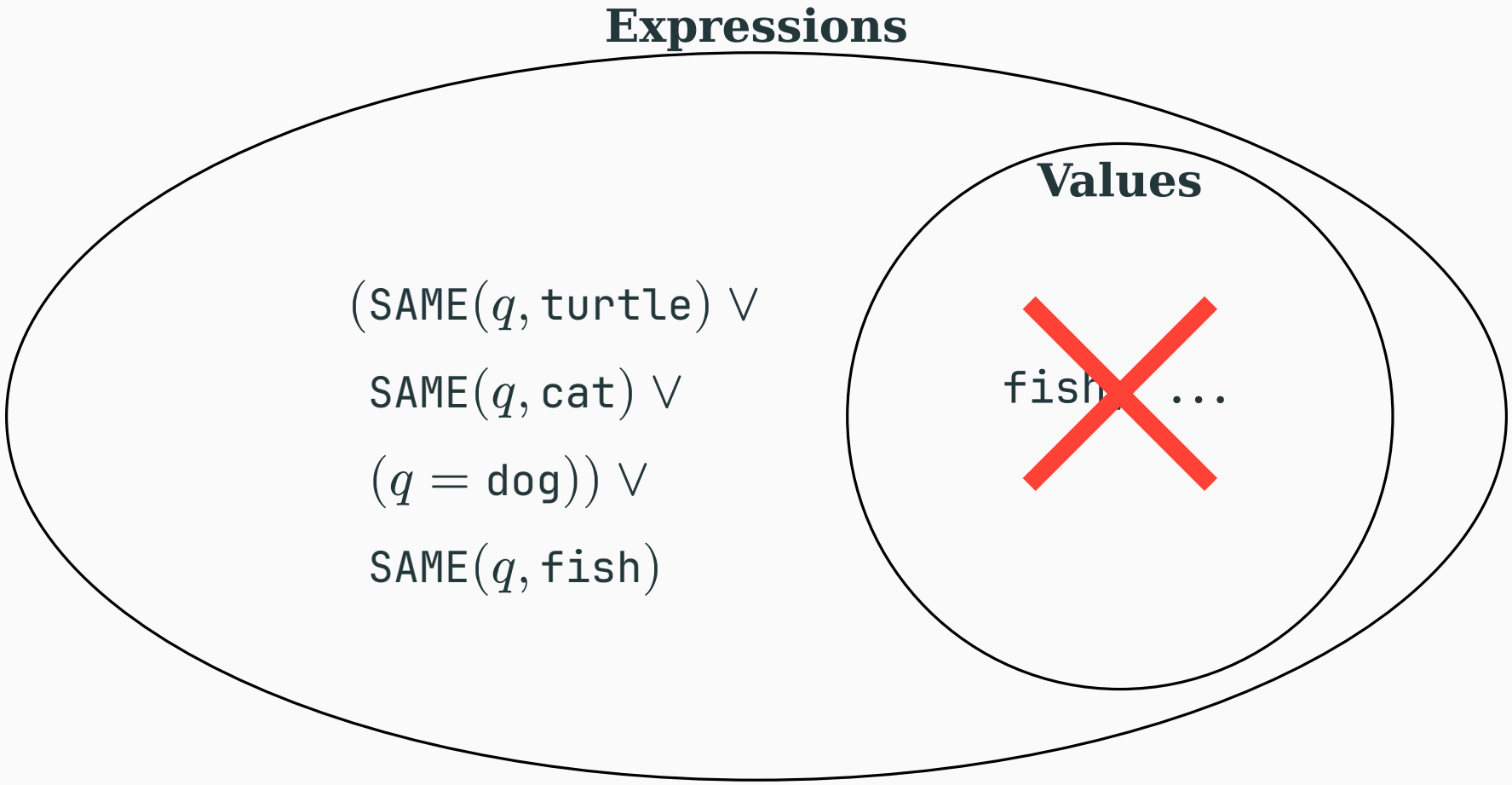

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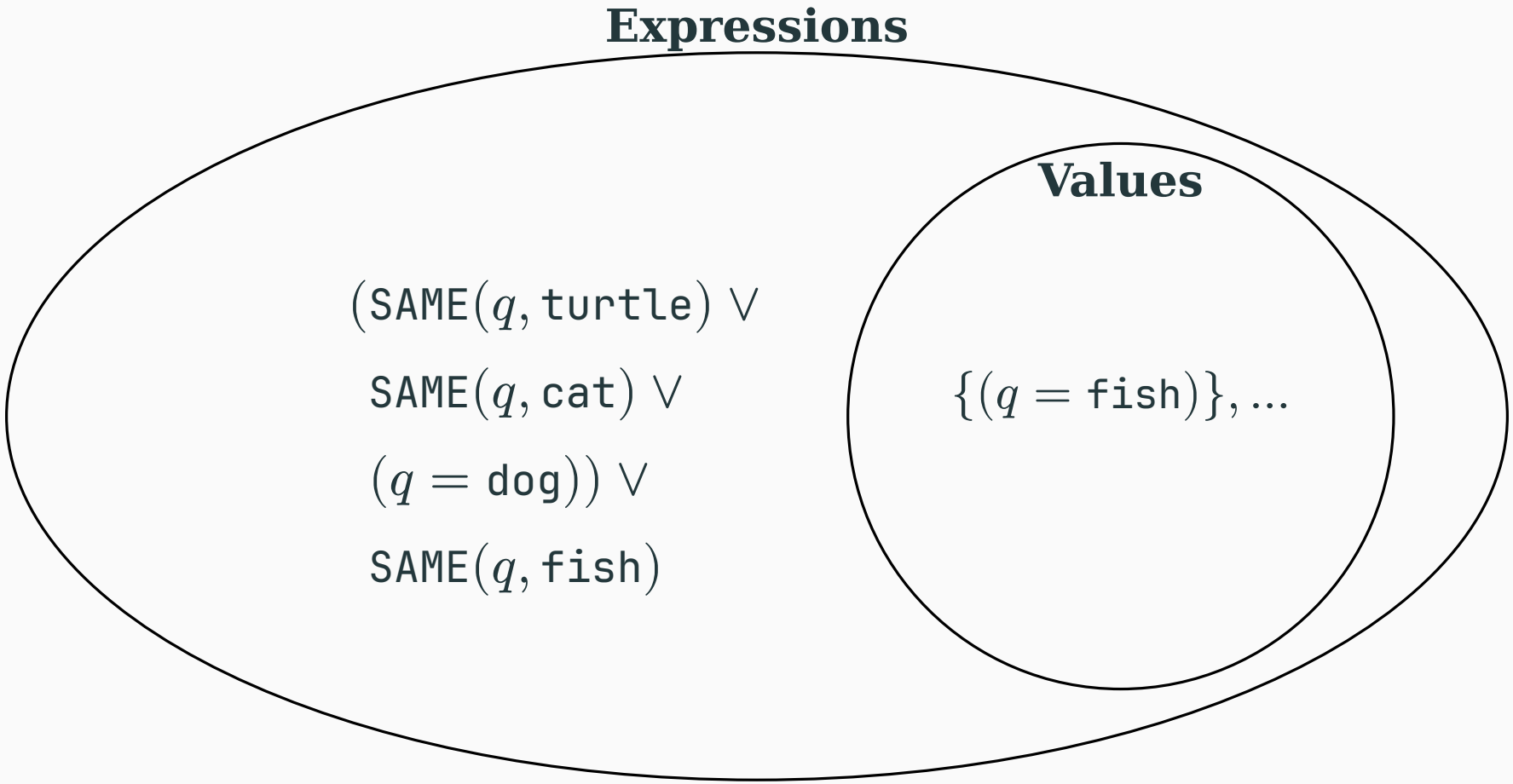

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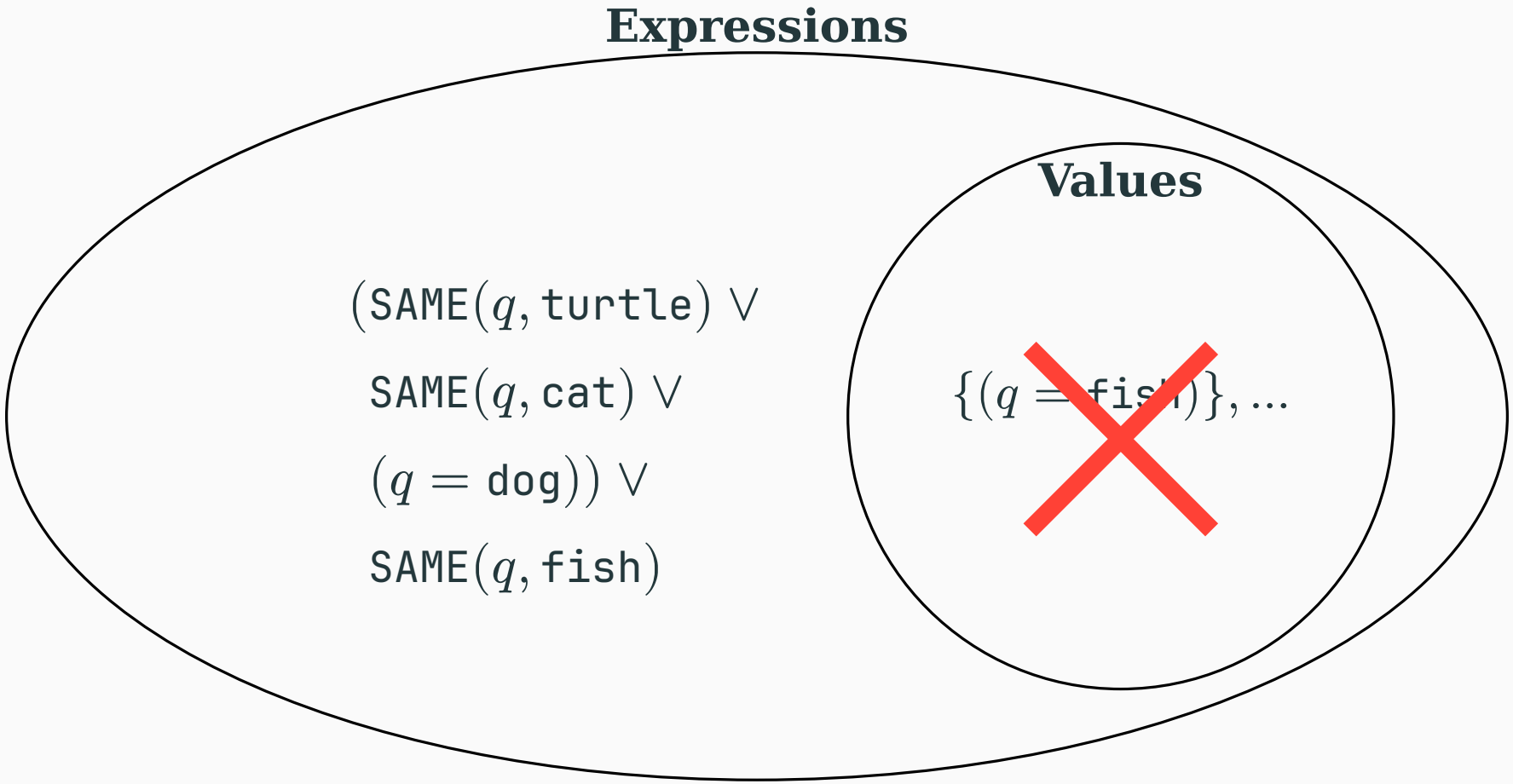

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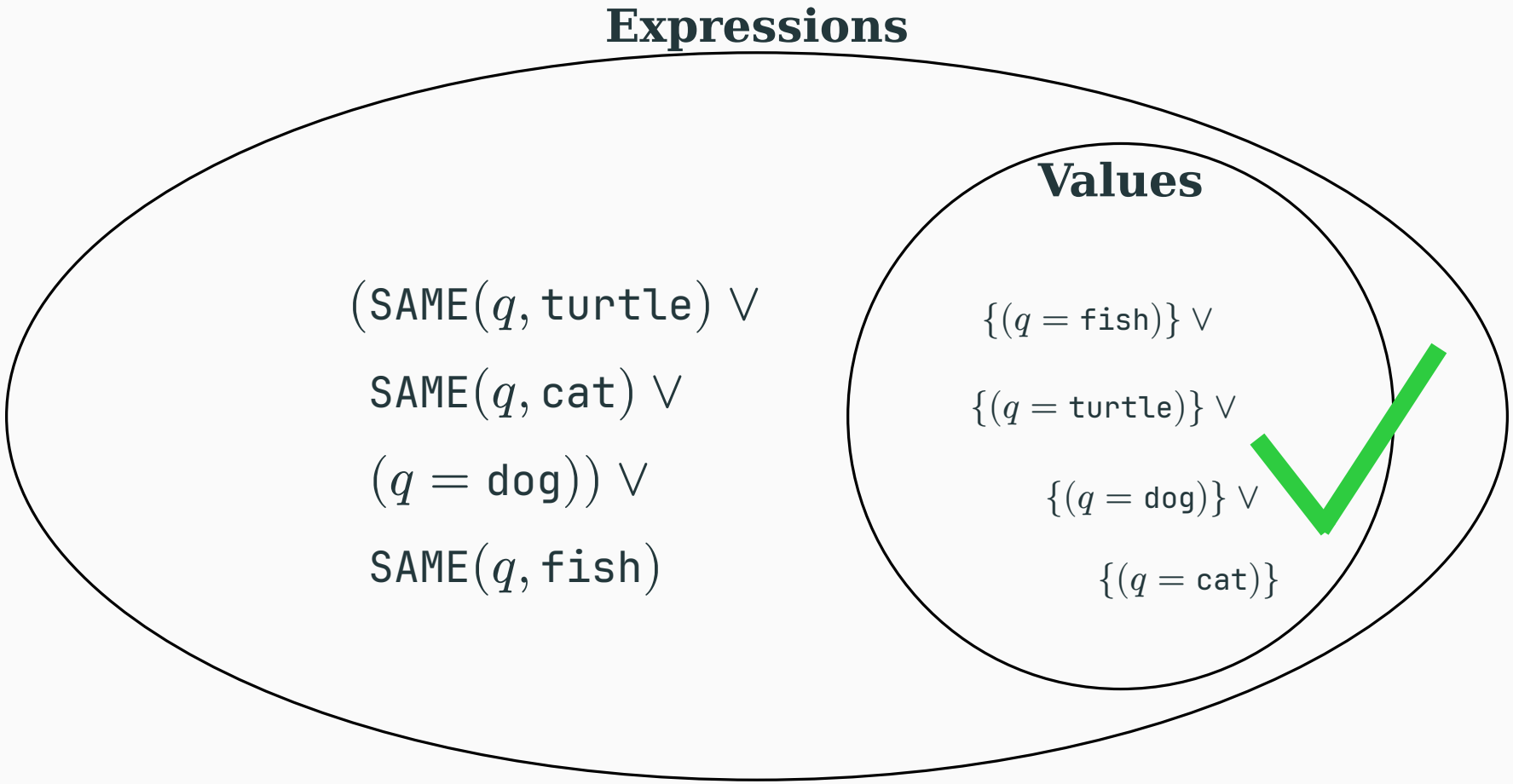

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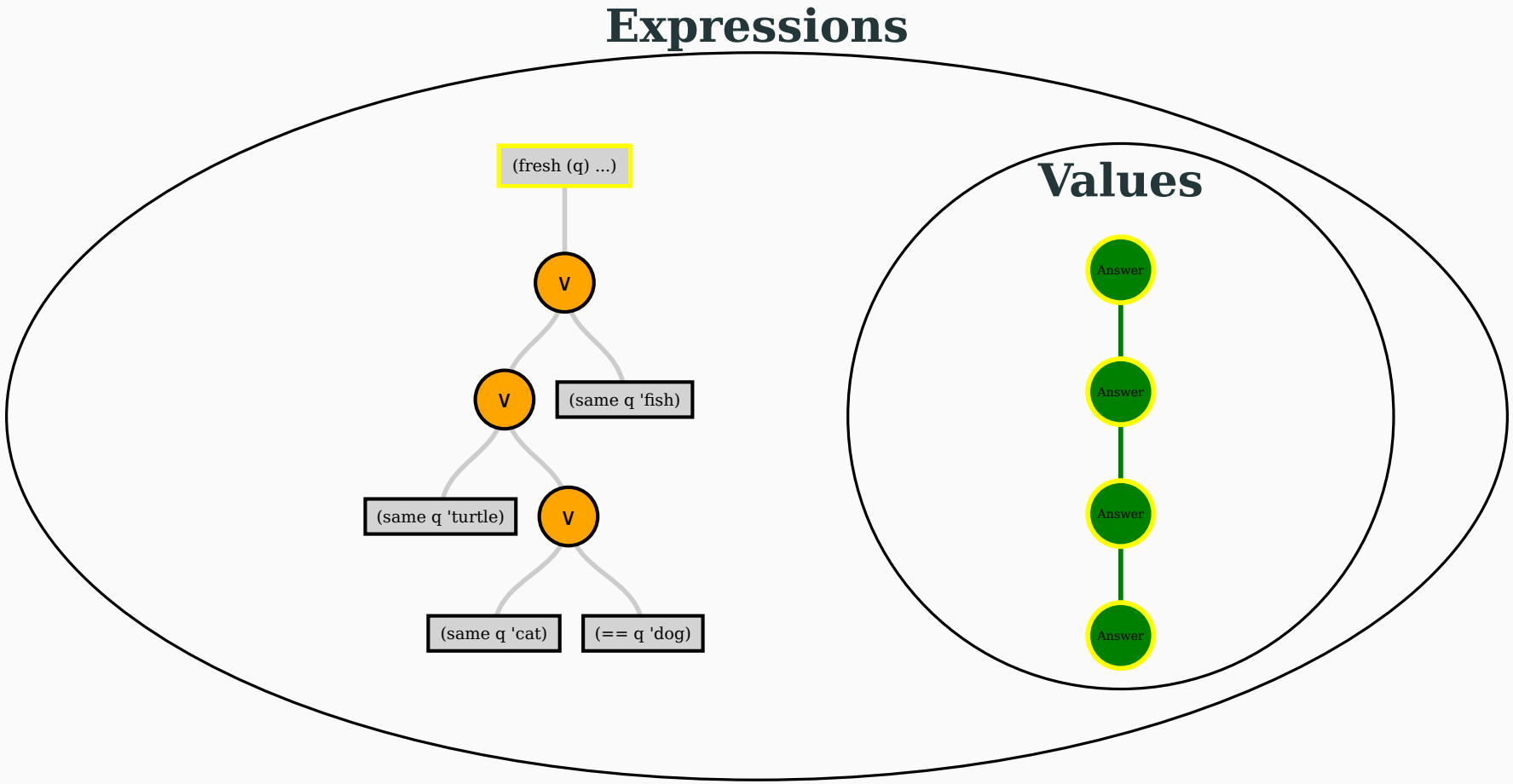


The values are normalized search trees!

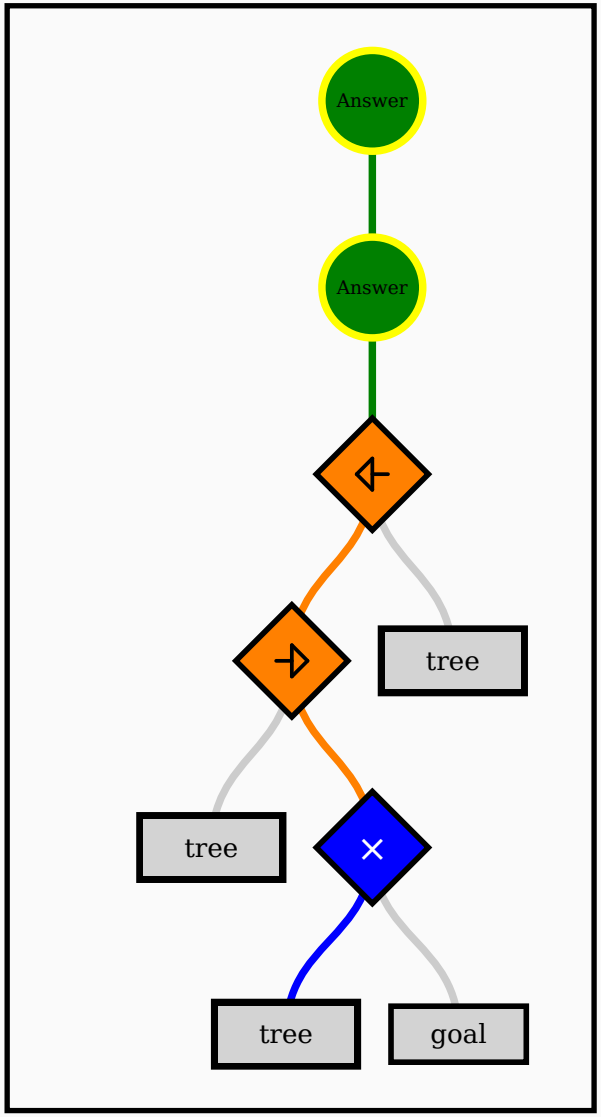
Contributions

Contributions

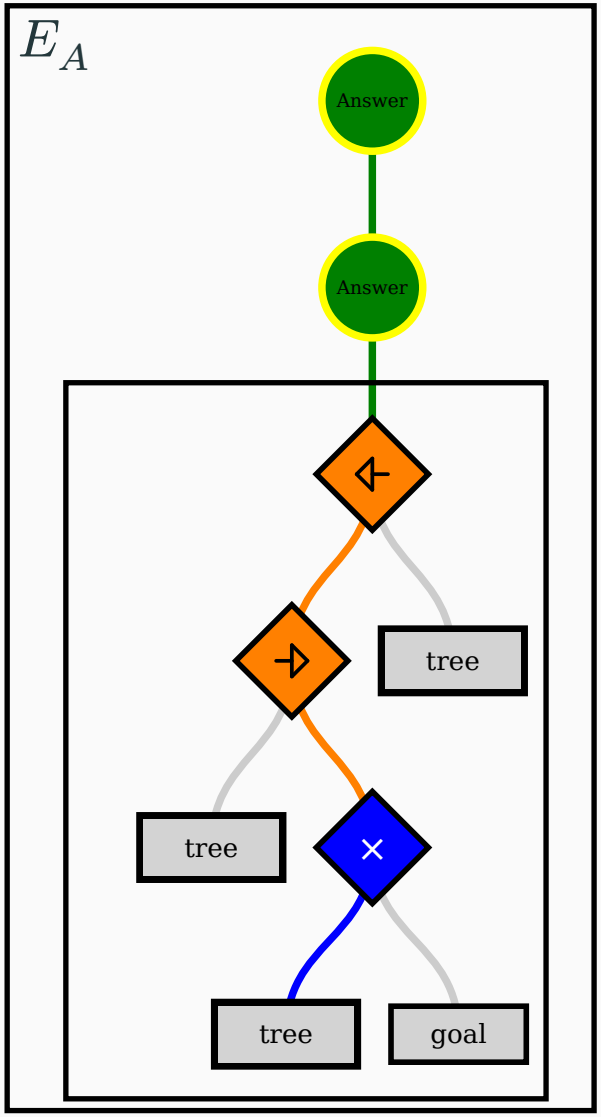
1. Small-Step Operational Semantics for miniKanren
   - Deterministic tree $\rightarrow$ tree rewrites
   - Mechanized and tested in PLT Redex
2. Tree based visual stepper
   - Executable semantics at the core
   - Implemented as a web app

Reduction Semantics

Evaluation Contexts

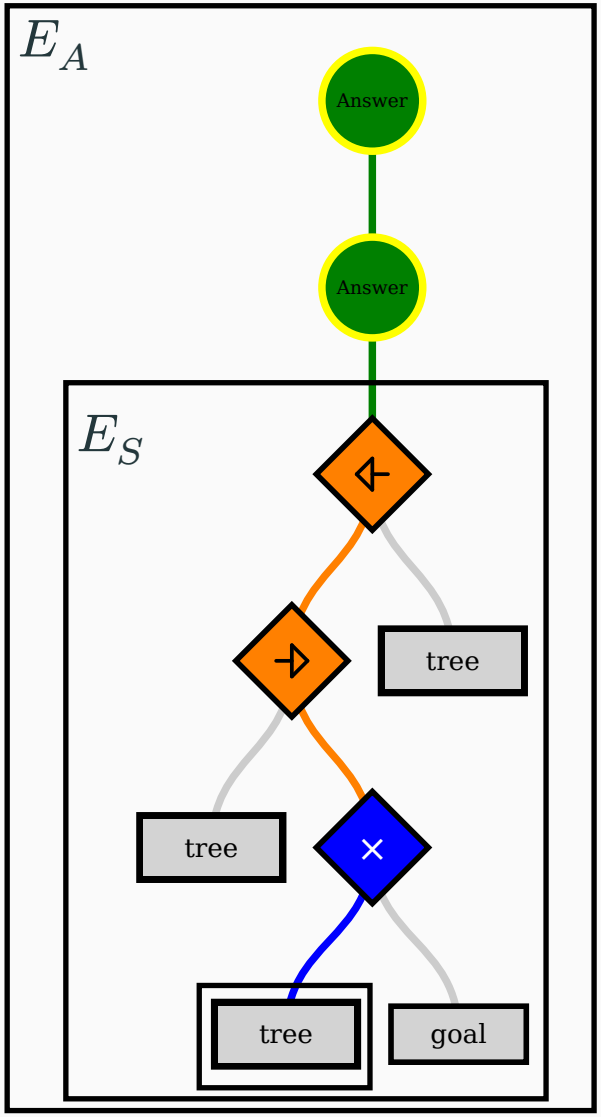


Evaluation Contexts



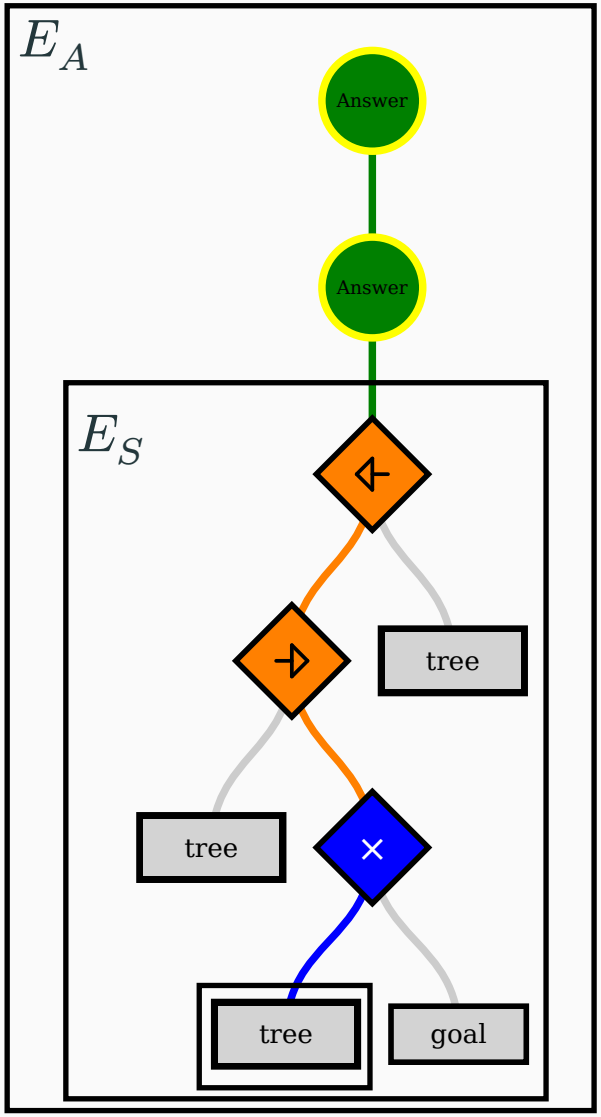
- E_A: Below answer stream

Evaluation Contexts



- E_A: Below answer stream
- E_S: Subtree within search tree

Evaluation Contexts

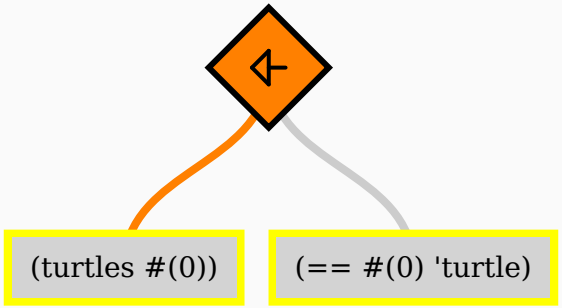


- E_A: Below answer stream
- E_S: Subtree within search tree
- $E = E_A[E_S\ \square]$

Turtles ... (DFS)

$$\texttt{TURTLES}(x) \leftarrow \texttt{TURTLES}(x) \vee (x = \texttt{turtle})$$

Turtles … (DFS)



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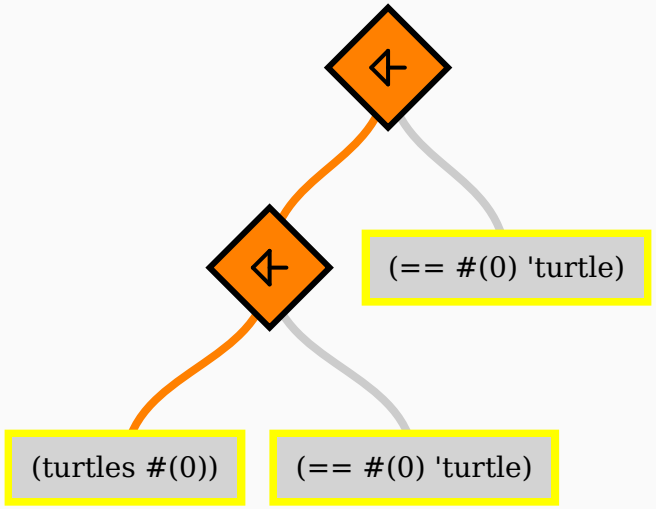

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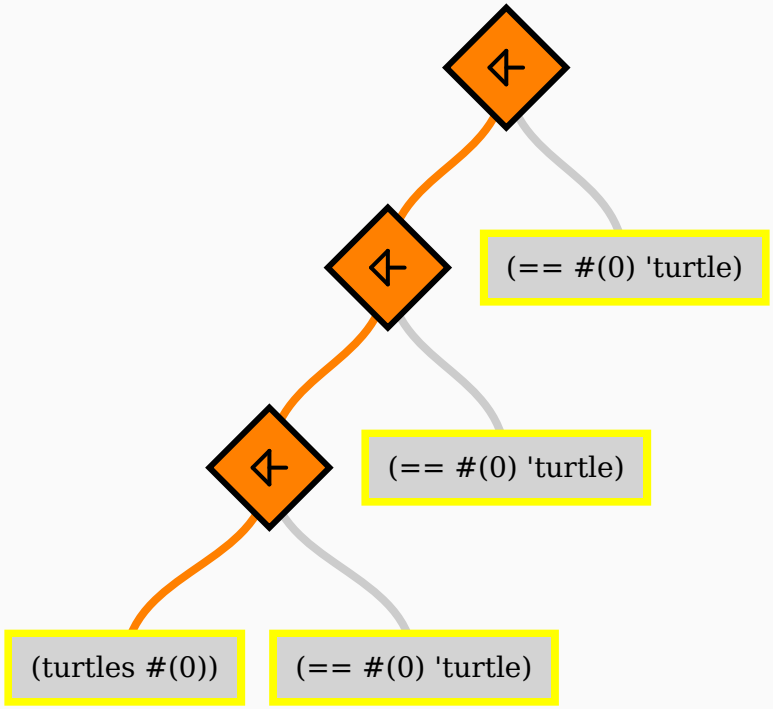

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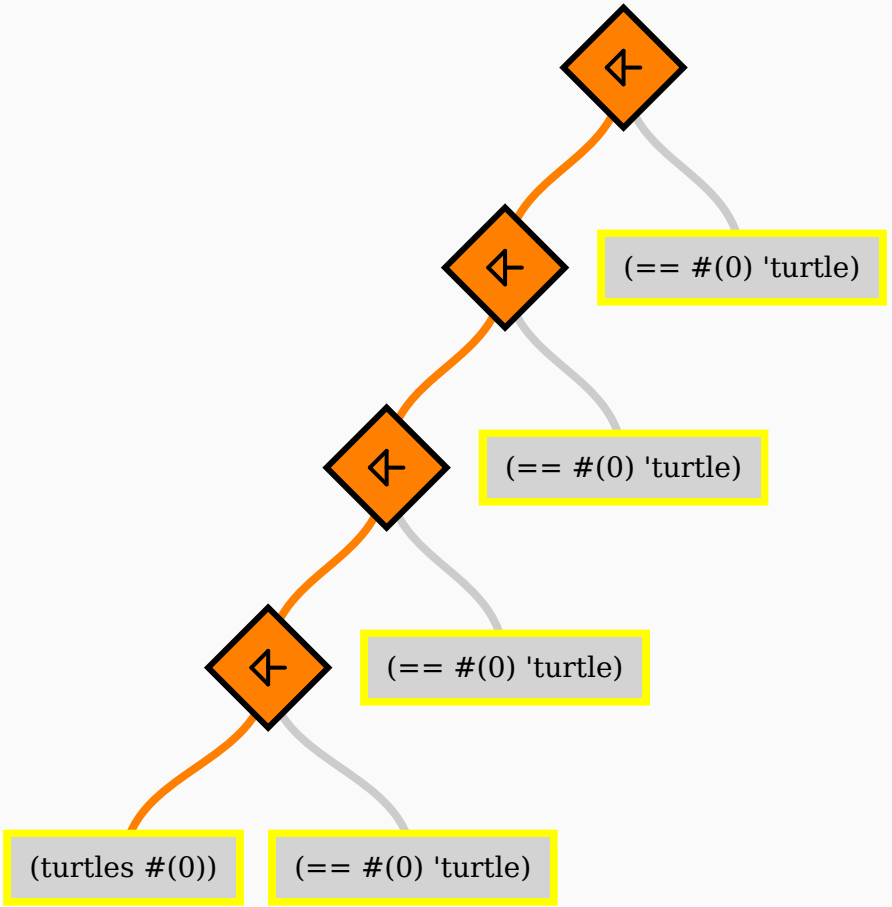

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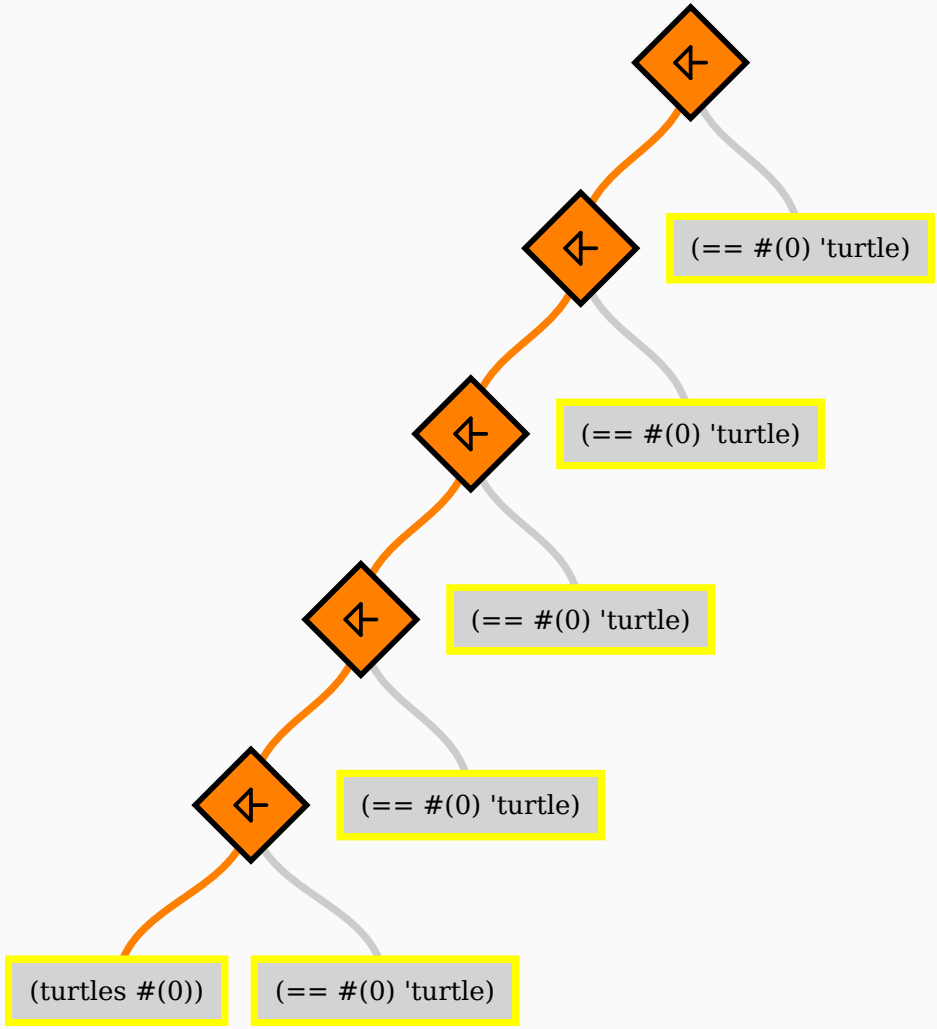
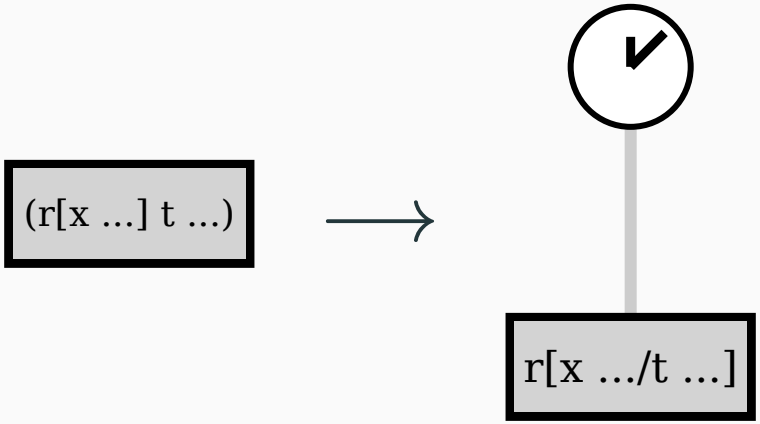
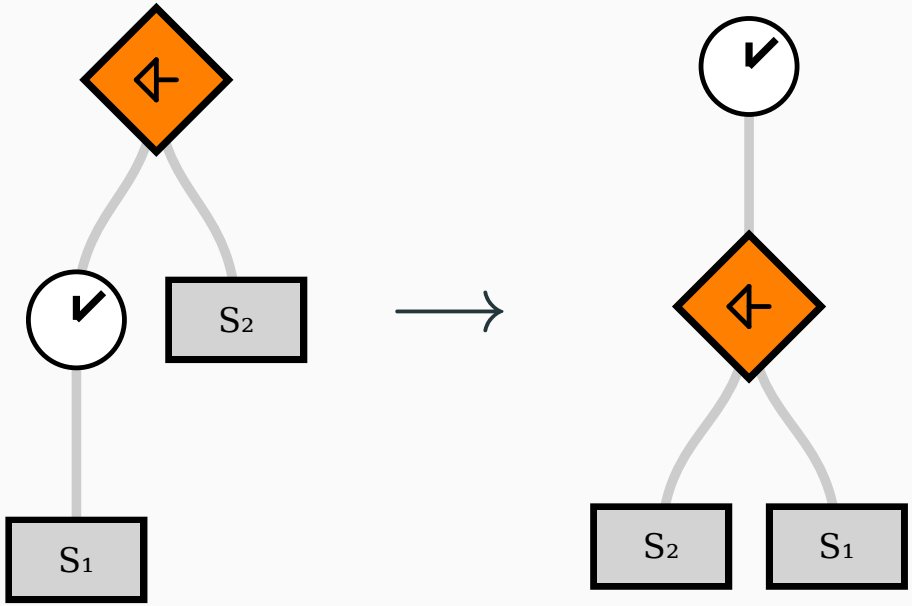


$$E[(r[x...]\ t...)\ \sigma] \longrightarrow E[\texttt{delay}\ \ r[x.../t...]\ \sigma]\ [\text{Delay}]$$

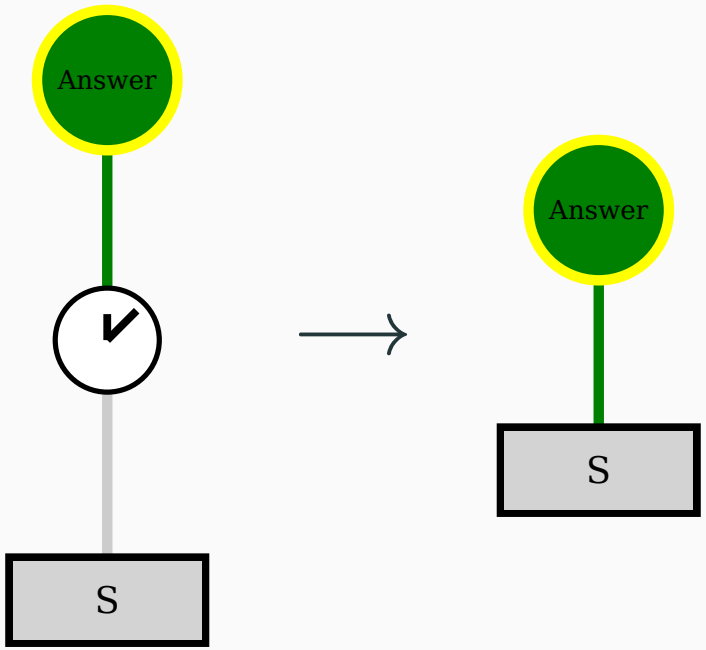


$$E[(\texttt{delay}\ S_1) \leftarrow S_2] \longrightarrow E[\texttt{delay}\ (S_2 \leftarrow S_1)]\ \text{[Interleave]}$$



Semantics :: Release Delay

$$E_A[(\texttt{delay}\ S)] \longrightarrow E_A[S]\ [\text{ReleaseDelay}]$$



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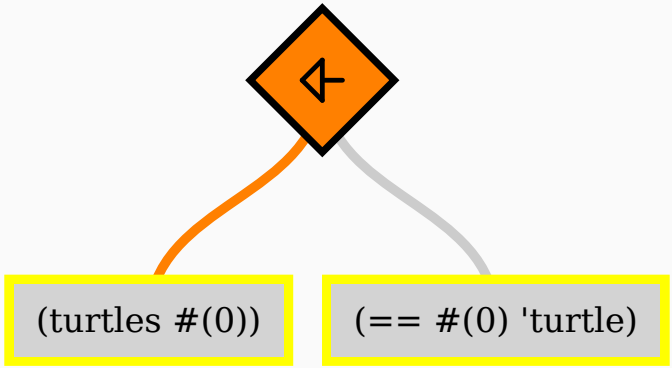

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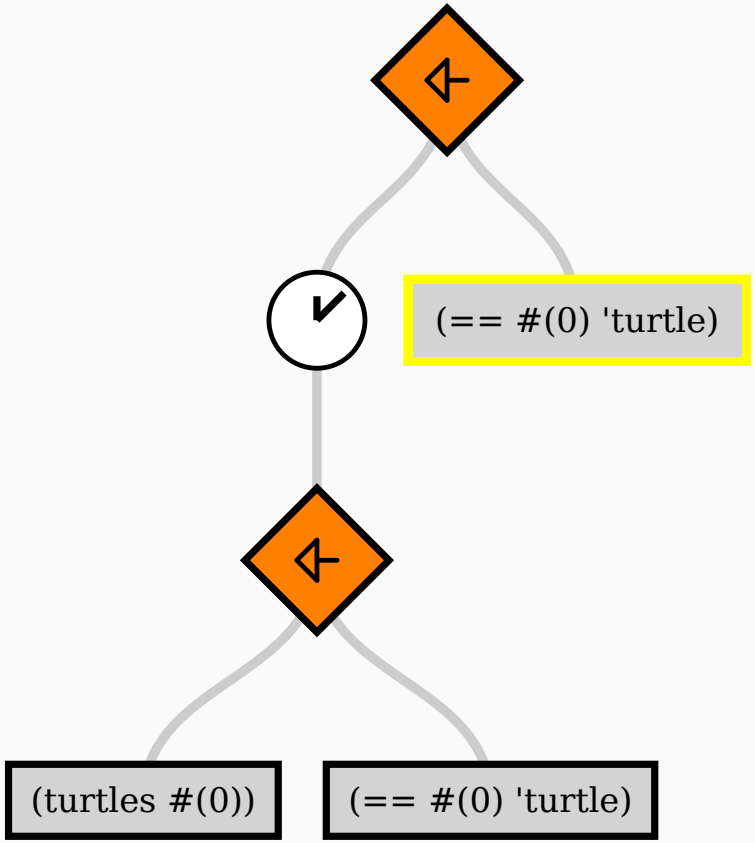

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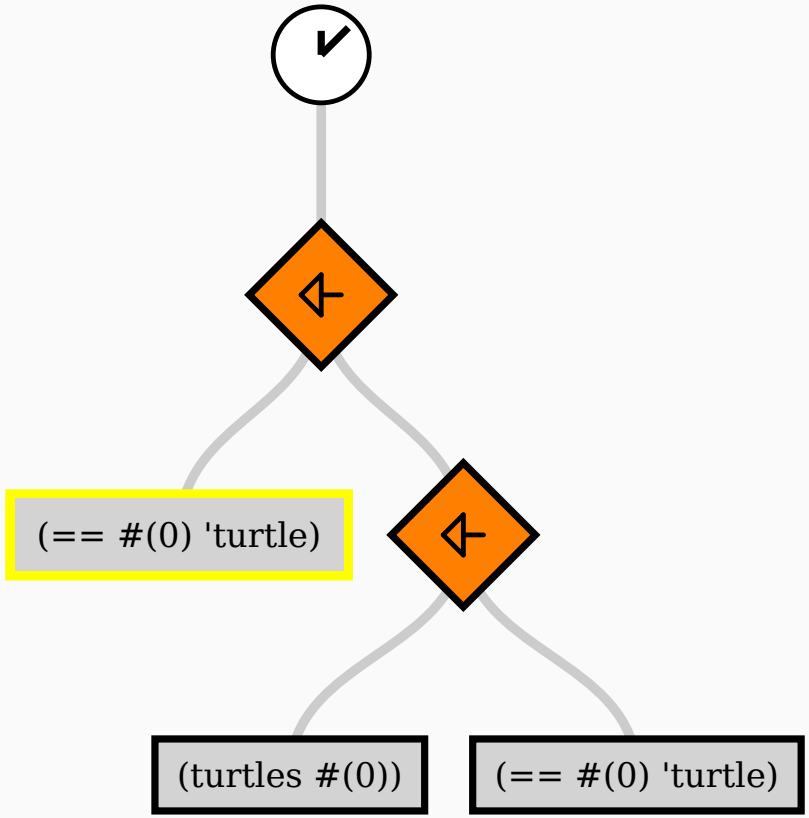

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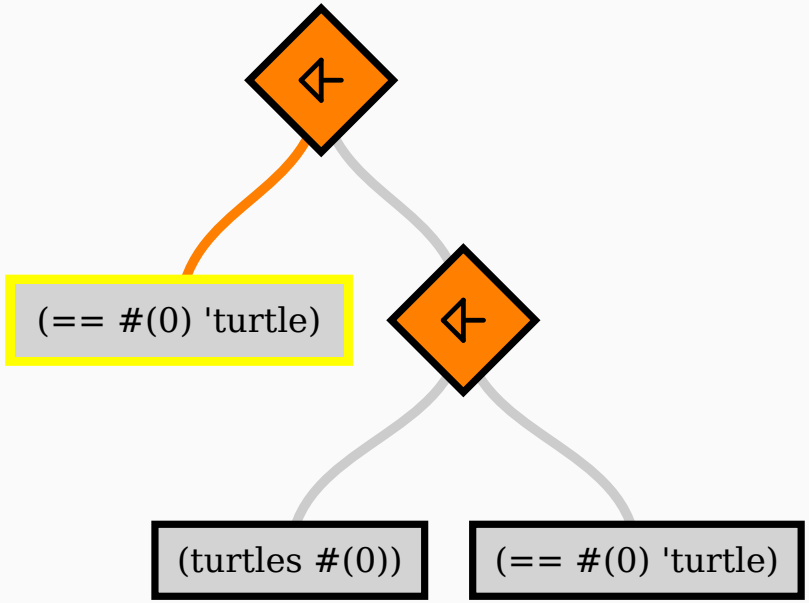

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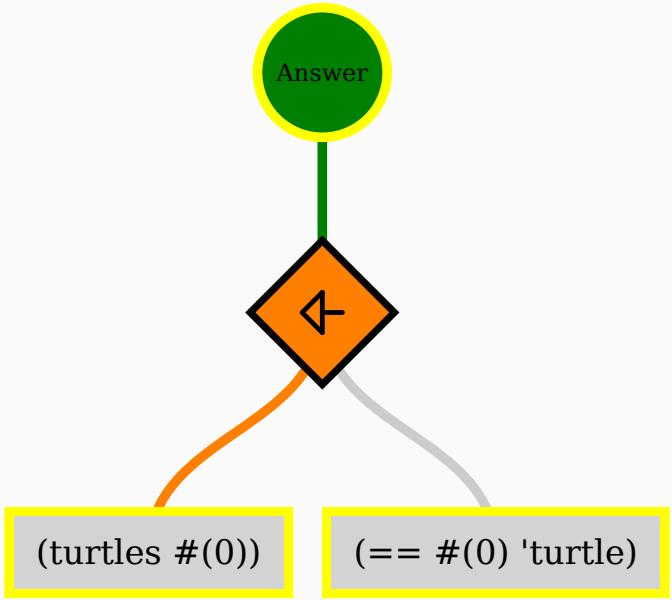

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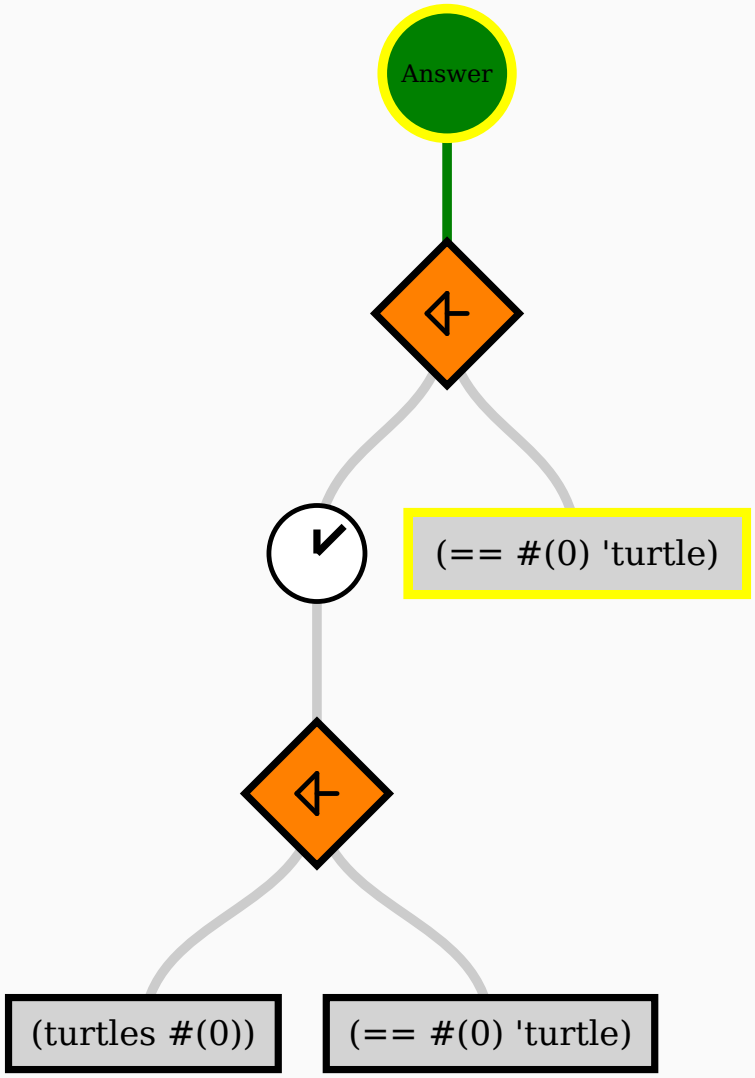

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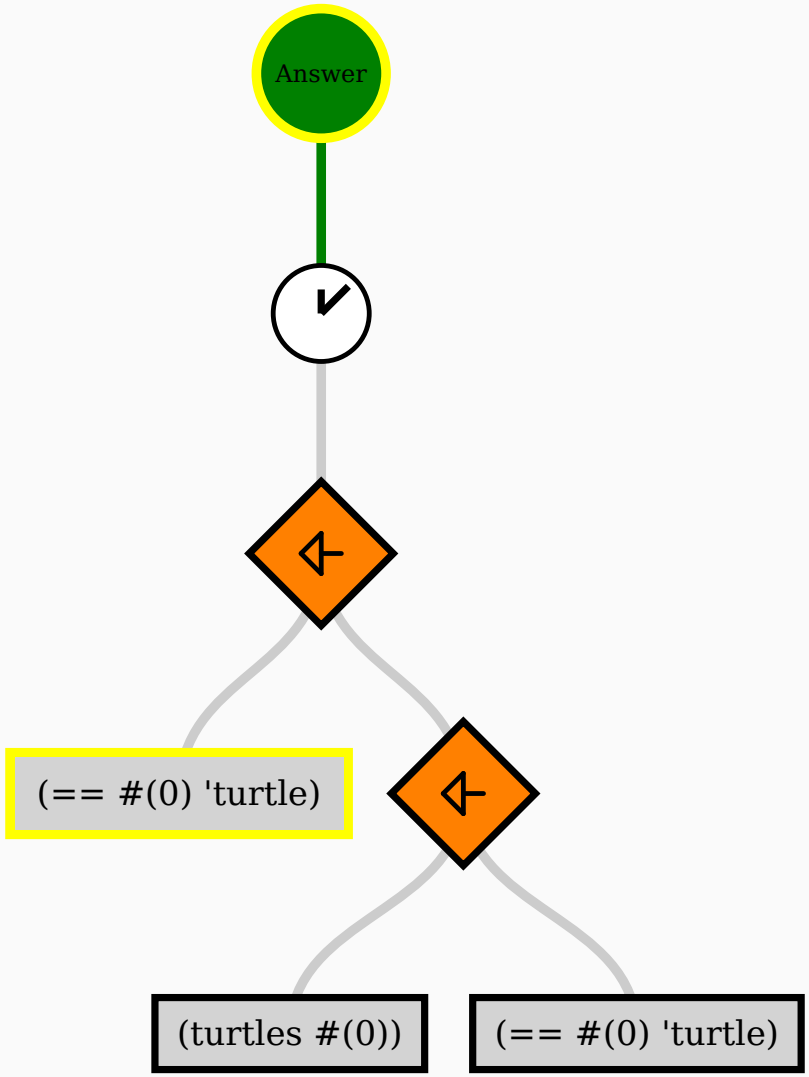

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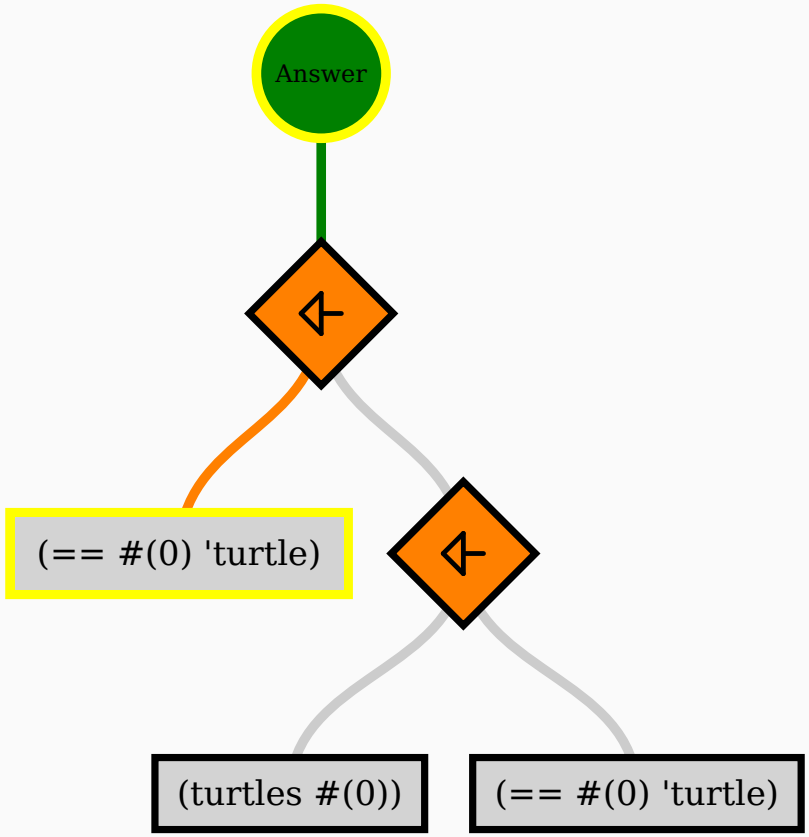

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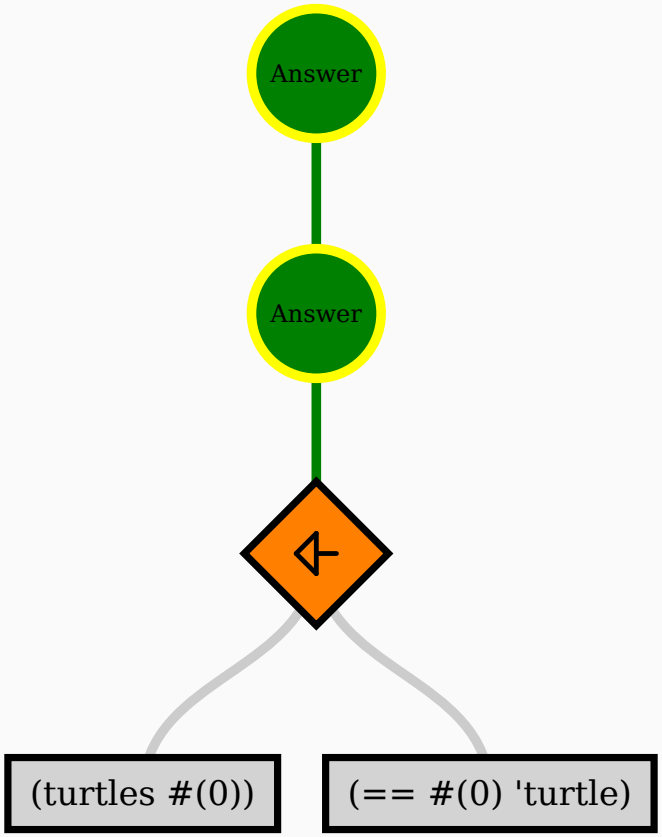


$$\text{Search Trees} \in S ::= S \leftarrow S$$

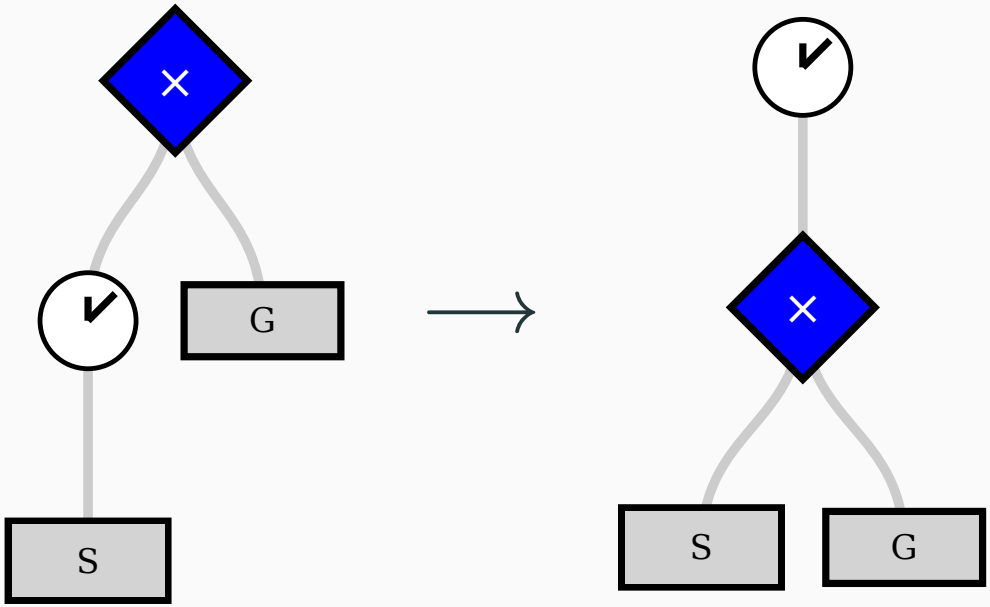
$$\mid \texttt{delay}\ S$$

$$\text{Search Trees} \in S ::= S \leftarrow S$$

$$\mid \texttt{delay}\ S$$

$$\mid S \times G$$

$$E[((\texttt{delay}\ S) \times G)] \longrightarrow E[(\texttt{delay}\ (S \times G))]\ [\text{DelayConj}]$$



$$\text{Search Trees} \in S ::= S \leftarrow S$$

$$\mid \texttt{delay}\ S$$

$$\mid S \times G$$

$$\mid G\ \sigma$$

Goal State

Goals

$$\text{Goals} \in G ::= r(x...) \qquad \text{Relation Call}$$

Goals

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$$\mid t = t \quad \text{Unification}$$

Goals

$$
\begin{aligned}
\text{Goals} \in G ::= {} & r(x\ldots) && \text{Relation Call} \\
| \; & t = t && \text{Unification} \\
| \; & \exists(x\ldots)G && \text{Fresh}
\end{aligned}
$$

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$$
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\mid {} & t = t && \text{Unification} \\
\mid {} & \exists(x\ldots)G && \text{Fresh} \\
\mid {} & G \vee G && \vee \\
\mid {} & G \wedge G && \wedge
\end{aligned}
$$

Search Trees :: Success and Failure

$$\text{Search Trees} \in S ::= S \leftarrow S$$

$$\mid \texttt{delay}\ S$$

$$\mid S \times G$$

$$\mid G\ \sigma$$

Goal State

$$\text{Search Trees} \in S ::= S \leftarrow S$$

$$\mid \texttt{delay}\ S$$

$$\mid S \times G$$

$$\mid G\,\sigma$$

Goal State

$$\mid \top\,\sigma$$

$$\text{Search Trees} \in S ::= S \leftarrow S$$

$$\mid \texttt{delay}\ S$$

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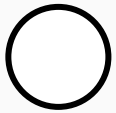
$$\mid G\,\sigma$$

Goal State

$$\mid \top\,\sigma$$

$$\mid \texttt{empty}$$



Answer Streams

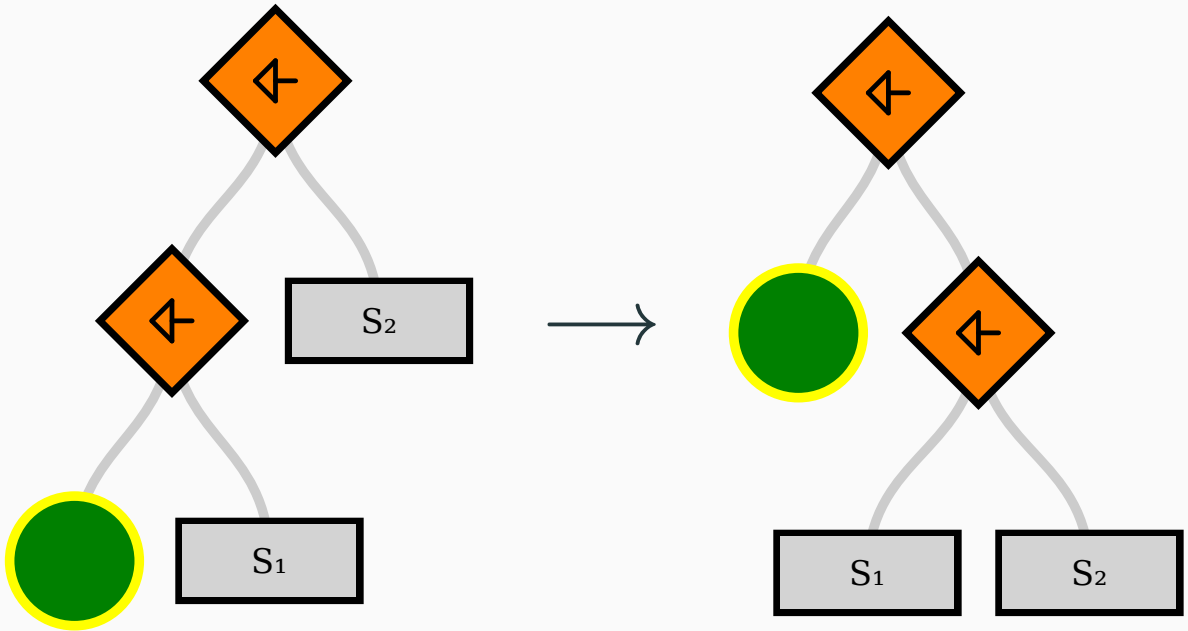
$$\text{Answer Streams} \in A ::= (\top\ \sigma) + A$$
$$\mid S$$

Semantics :: Goal Evaluation

$$E[(G\ \sigma)] \longrightarrow E[(\top \sigma)]\ \text{[GoalSucc]}$$

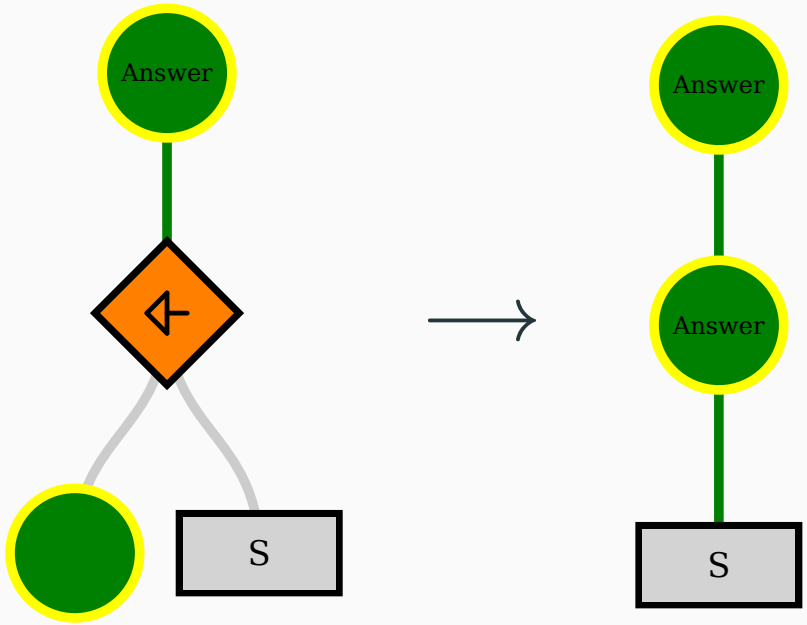
$$E[(G\ \sigma)] \longrightarrow E[\texttt{empty}]\ \text{[GoalFail]}$$

$$E[((\top\ \sigma) \leftarrow S_1) \leftarrow S_2] \longrightarrow E[(\top\ \sigma) \leftarrow (S_1 \leftarrow S_2)]\ \text{[ReassocDisj]}$$



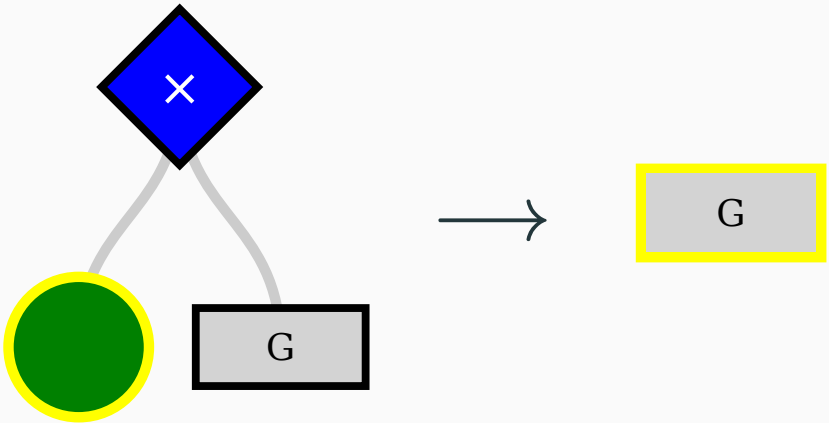
Semantics :: Promote

$$E_A[(\top\ \sigma) \leftarrow S] \longrightarrow E_A[(\top\ \sigma) + S]\ \text{[Promote]}$$

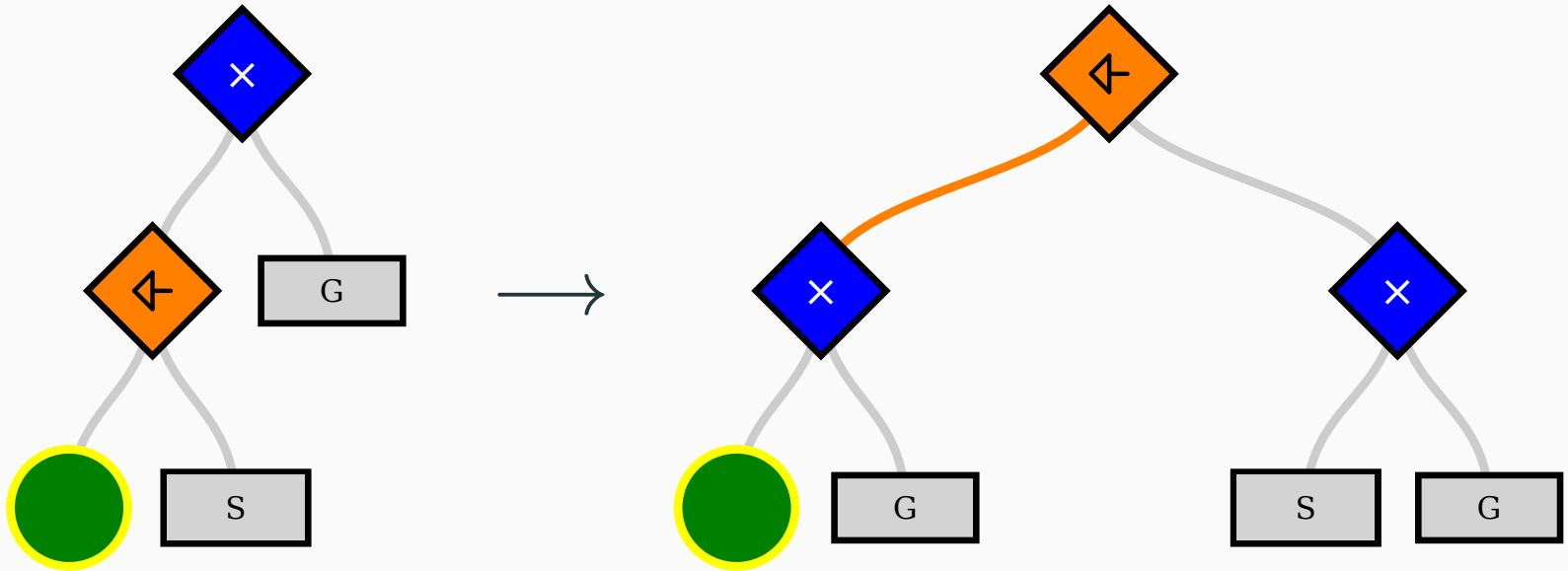


Semantics :: SuccConj

$$E[(\top\ \sigma) \times G] \longrightarrow E[G\ \sigma]\ \text{[SuccConj]}$$

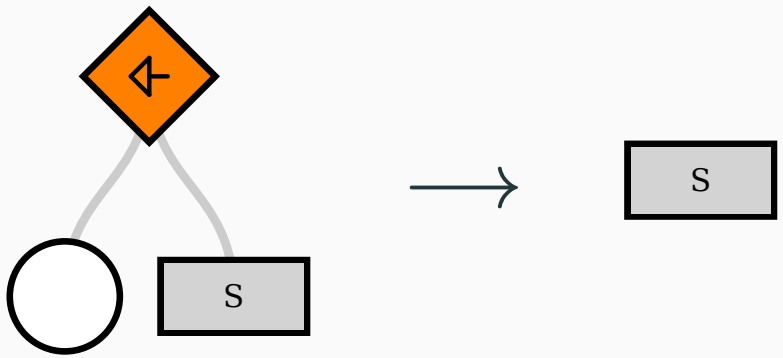


$$E[((\top\ \sigma) \leftarrow S) \times G] \longrightarrow E[((\top\ \sigma) \times G) \leftarrow (S \times G)]\ \text{[DistrConj]}$$

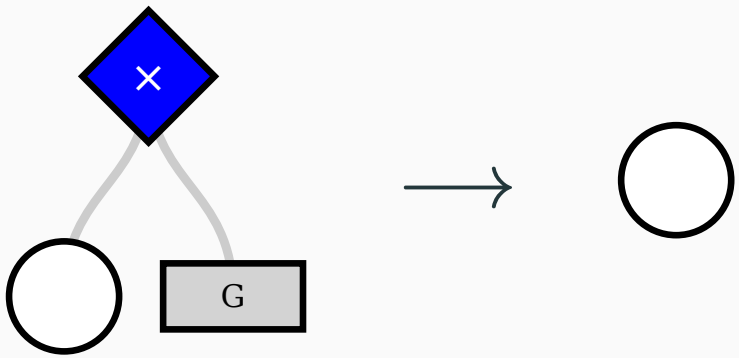


Semantics :: Pruning

$E[\texttt{empty} \leftarrow S] \longrightarrow E[S]$ [PruneDisj]



$E[\texttt{empty} \times G] \longrightarrow E[\texttt{empty}]$ [PruneConj]



Validation

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- **Syntactic Preservation**
  - Every reduction step preserves well-formedness.
    - Valid substitution structure
    - Properly scoped logic variables
    - Relation existence and arity

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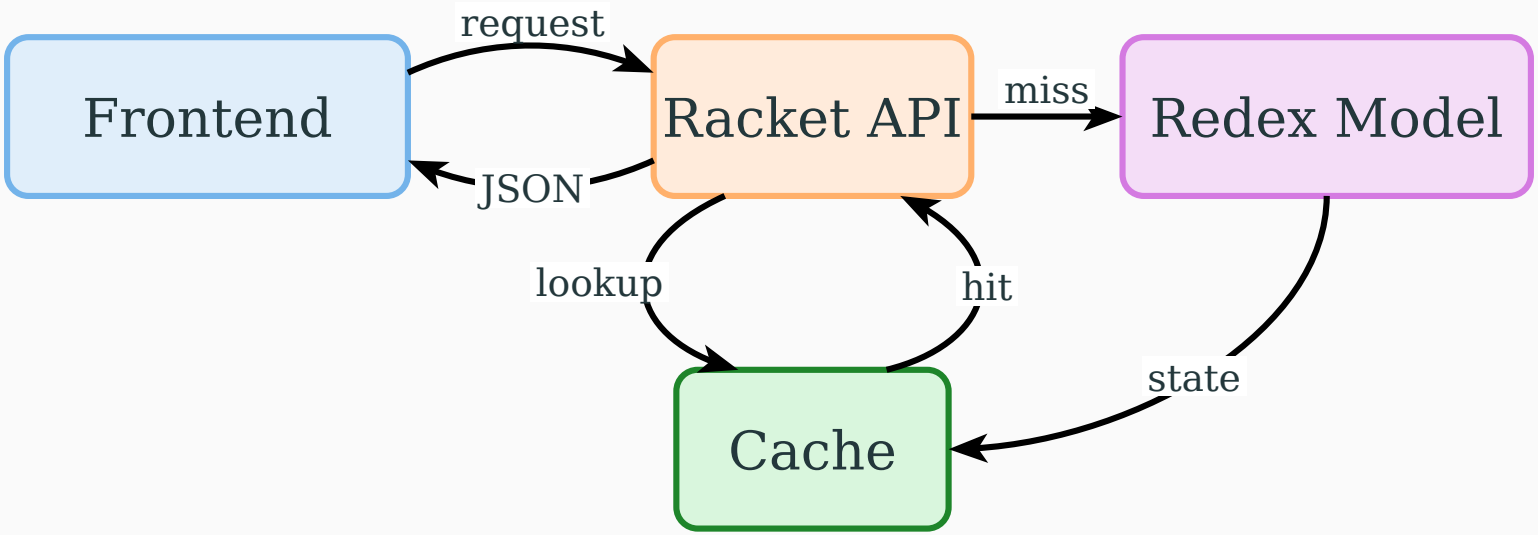
  - Every reduction step preserves well-formedness.
    - Valid substitution structure
    - Properly scoped logic variables
    - Relation existence and arity
- **Determinism**
  - Every well-formed program can take at most one step
- **Monotonic Propagation of Delayed Subtrees**
  - Delayed subtrees cannot become stranded
  - Resumption occurs at the top of the search tree

Visualizer

Visualizer :: Architecture



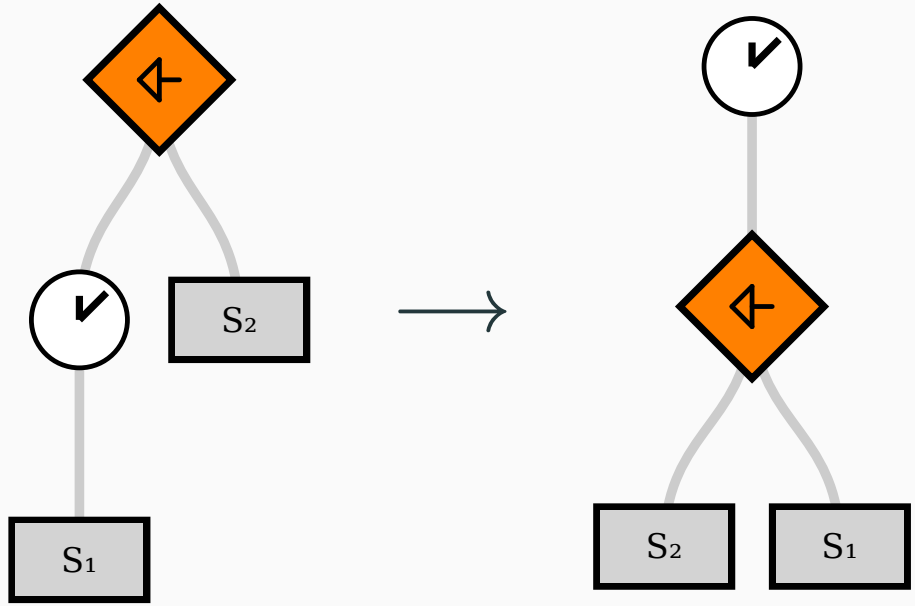
UX Enhanced Model

1. Railway Model (Directed Disjunctions)
2. Lazy Relation Expansion
3. Tagging for Bidirectional Tracing
4. Trail (History of Equations)
5. Current Reification/Projection

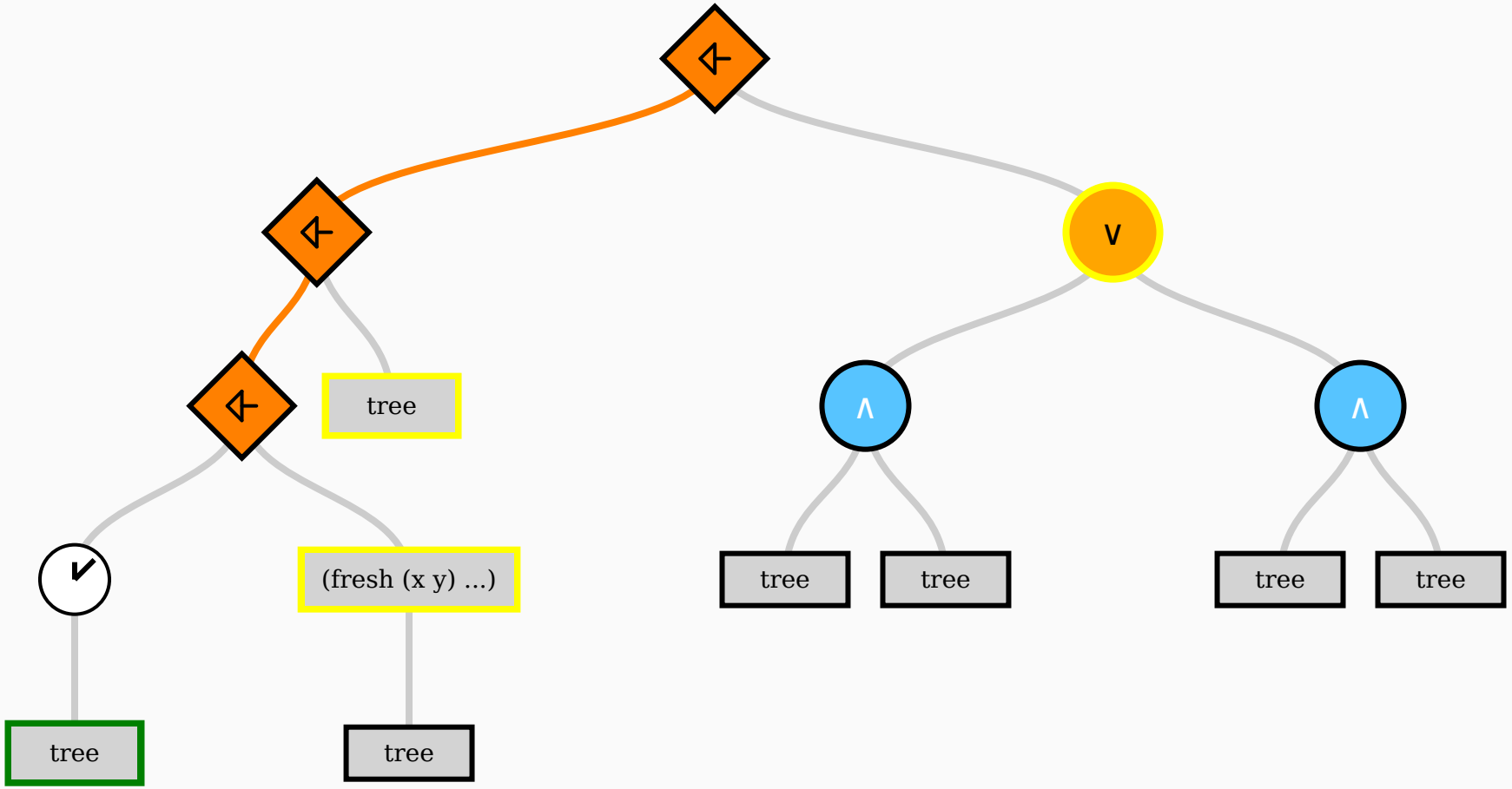
Railway Model

Old Interleaving

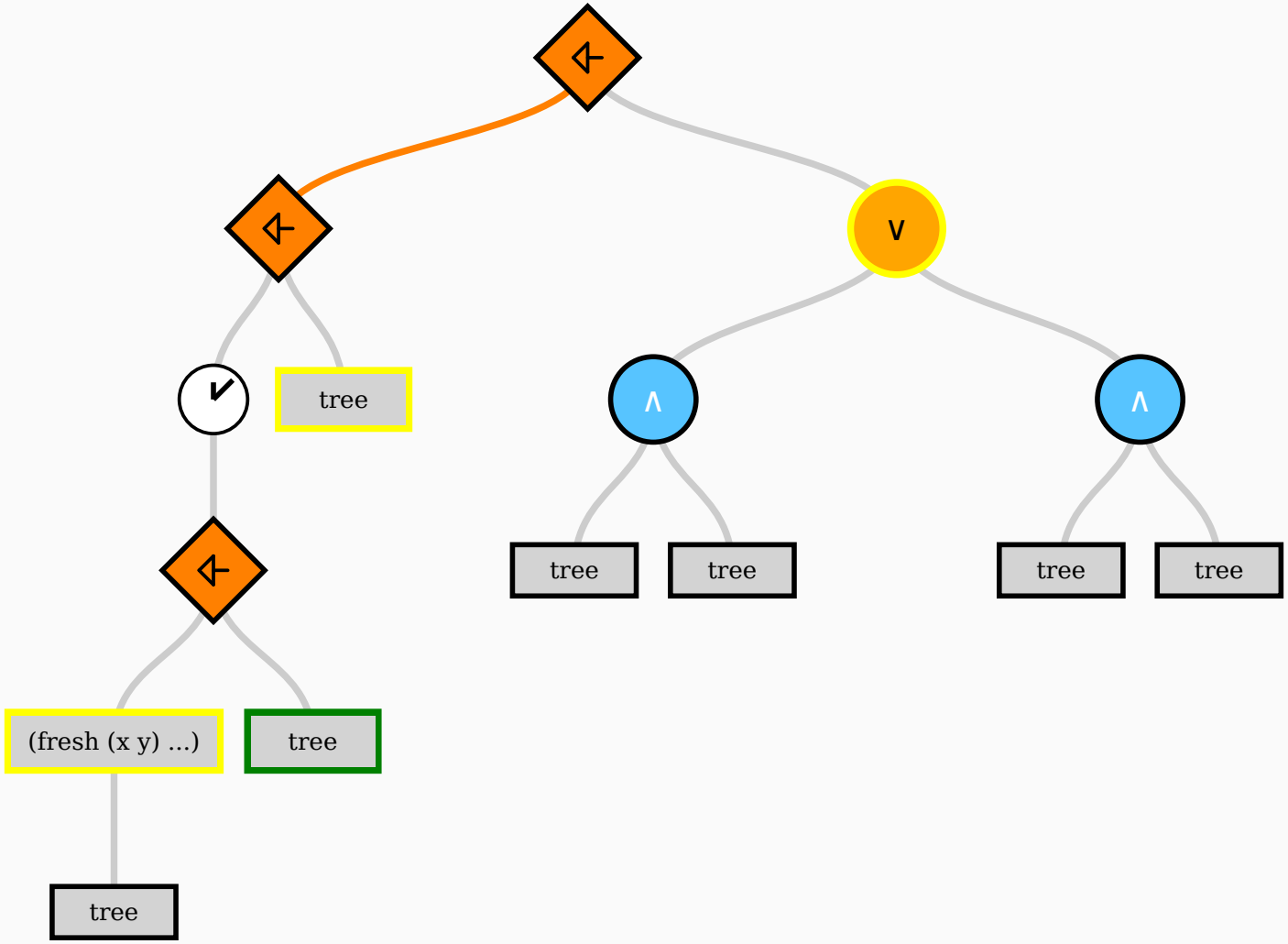
$$E[(\texttt{delay}\ S_1) \leftarrow S_2] \longrightarrow E[\texttt{delay}\ (S_2 \leftarrow S_1)]\ [\text{Interleave (Old)}]$$



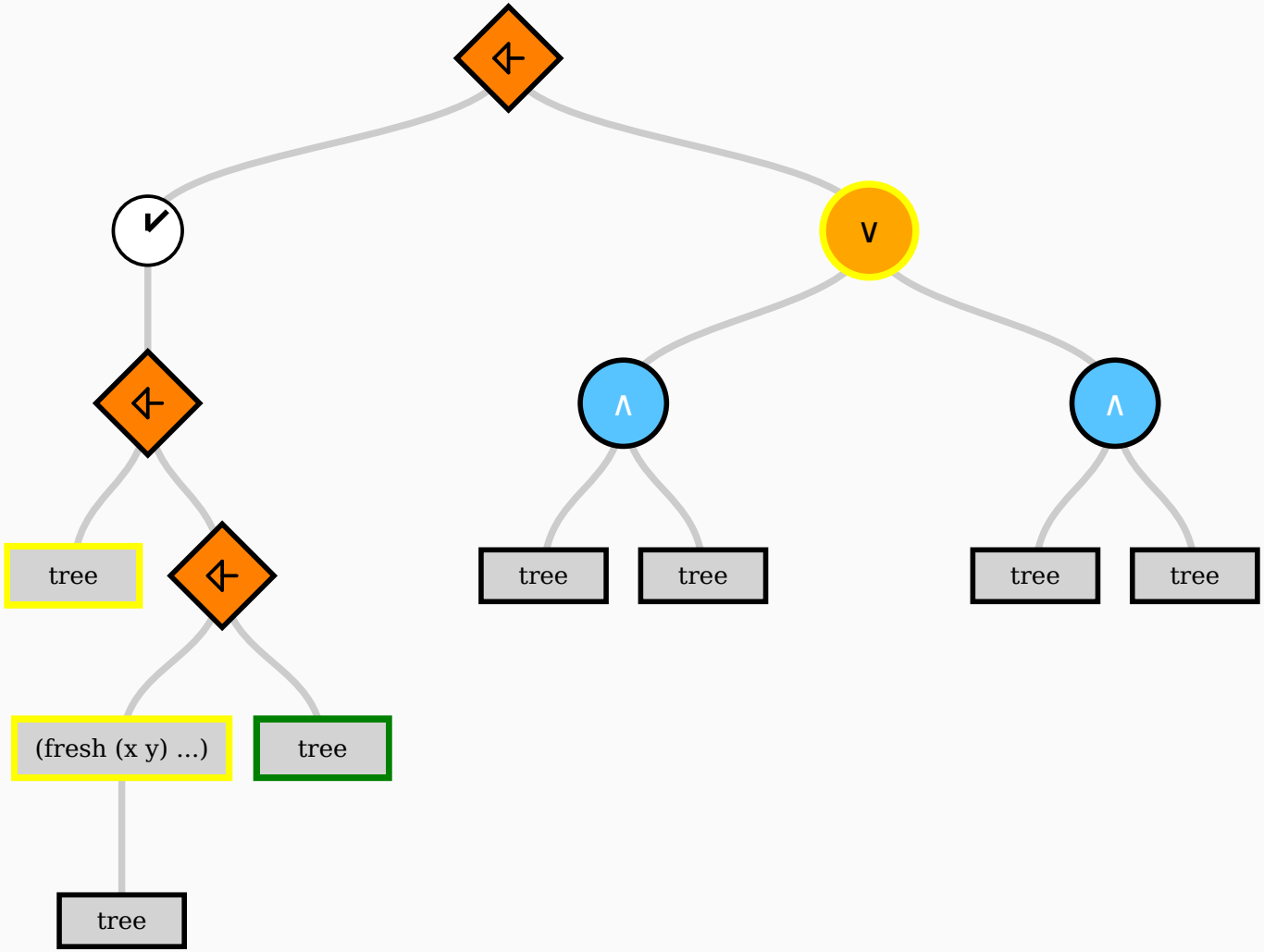
Old Interleaving



Old Interleaving



Old Interleaving



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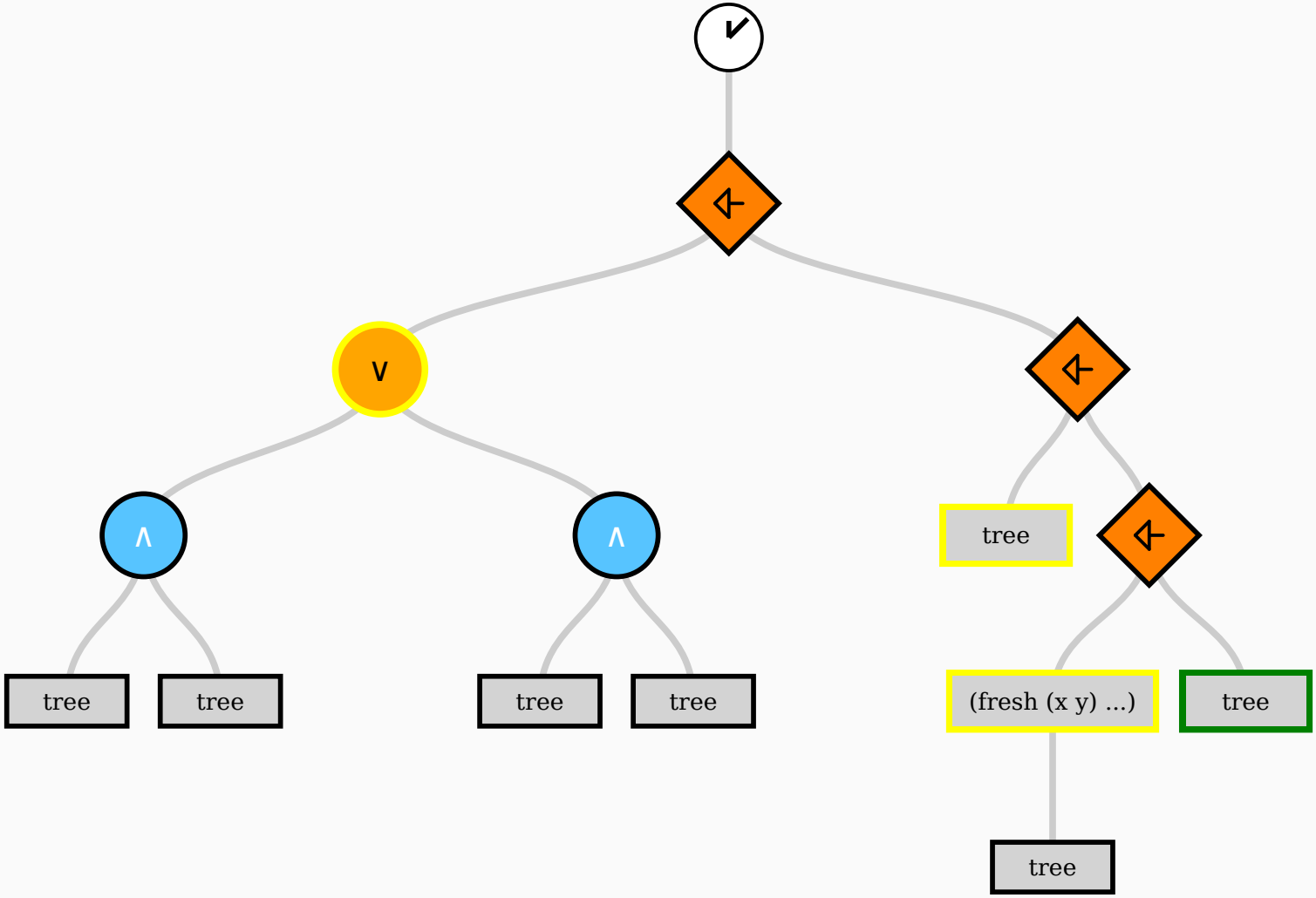

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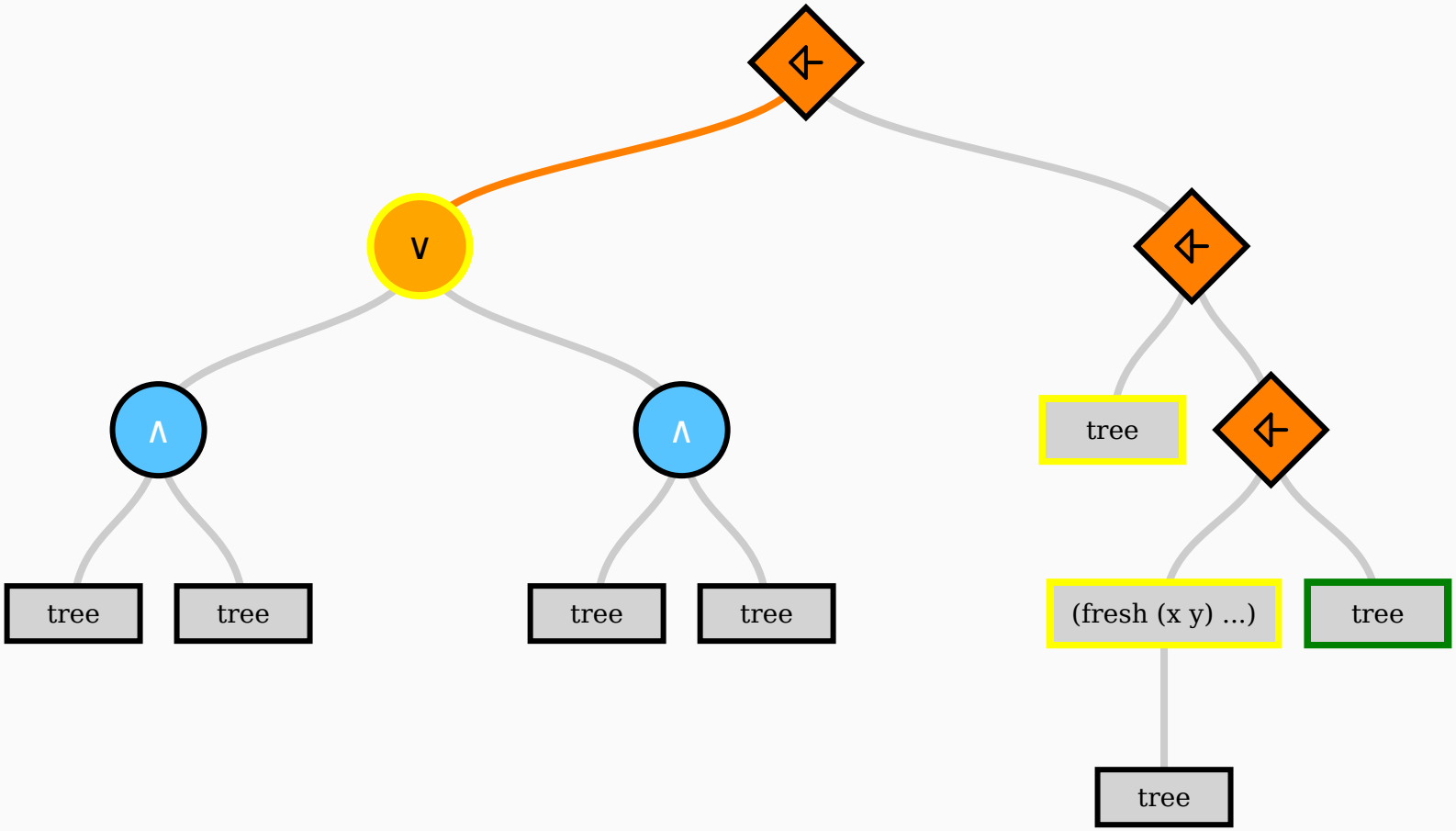
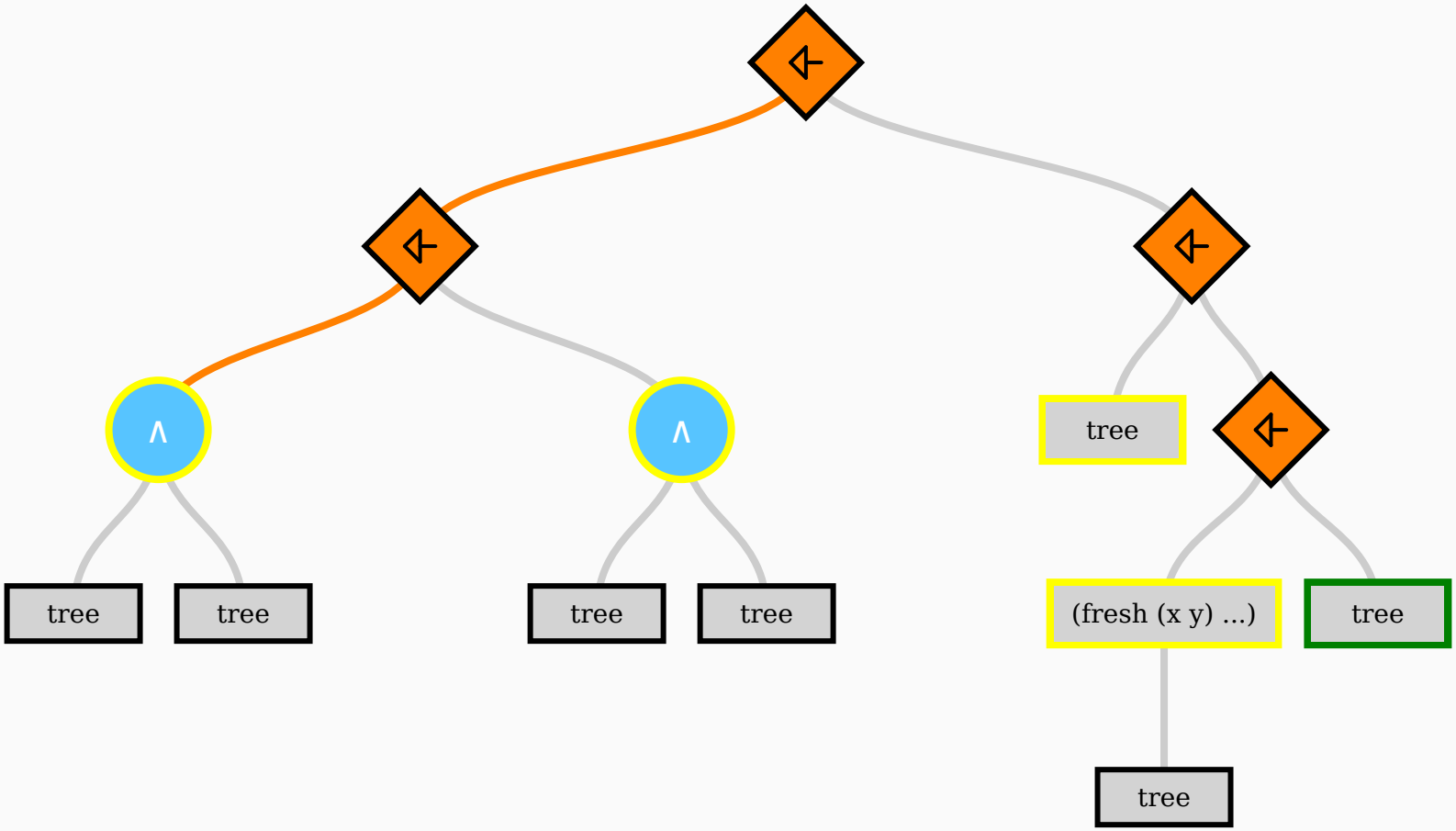
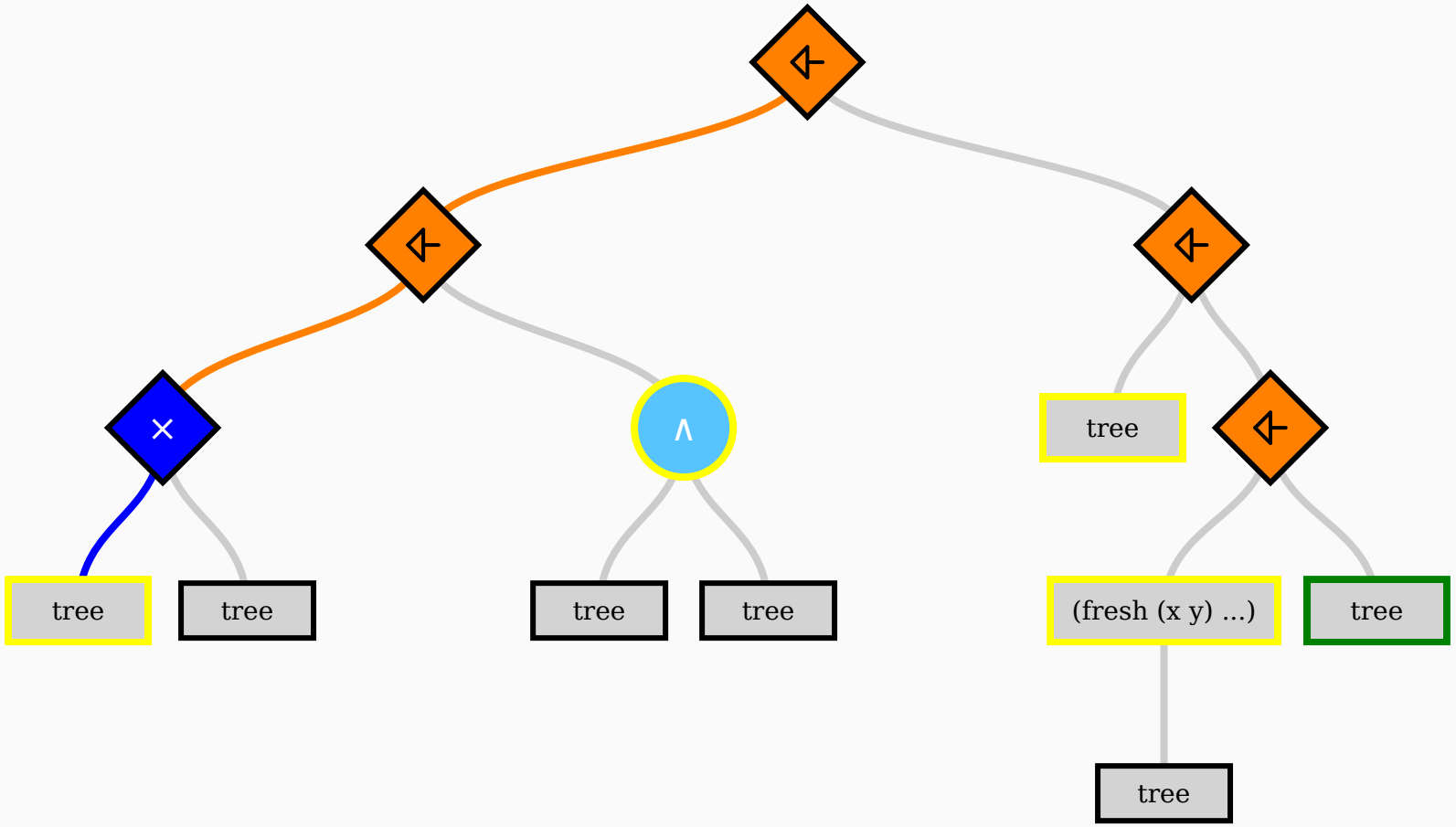


Old Interleaving



Old Interleaving



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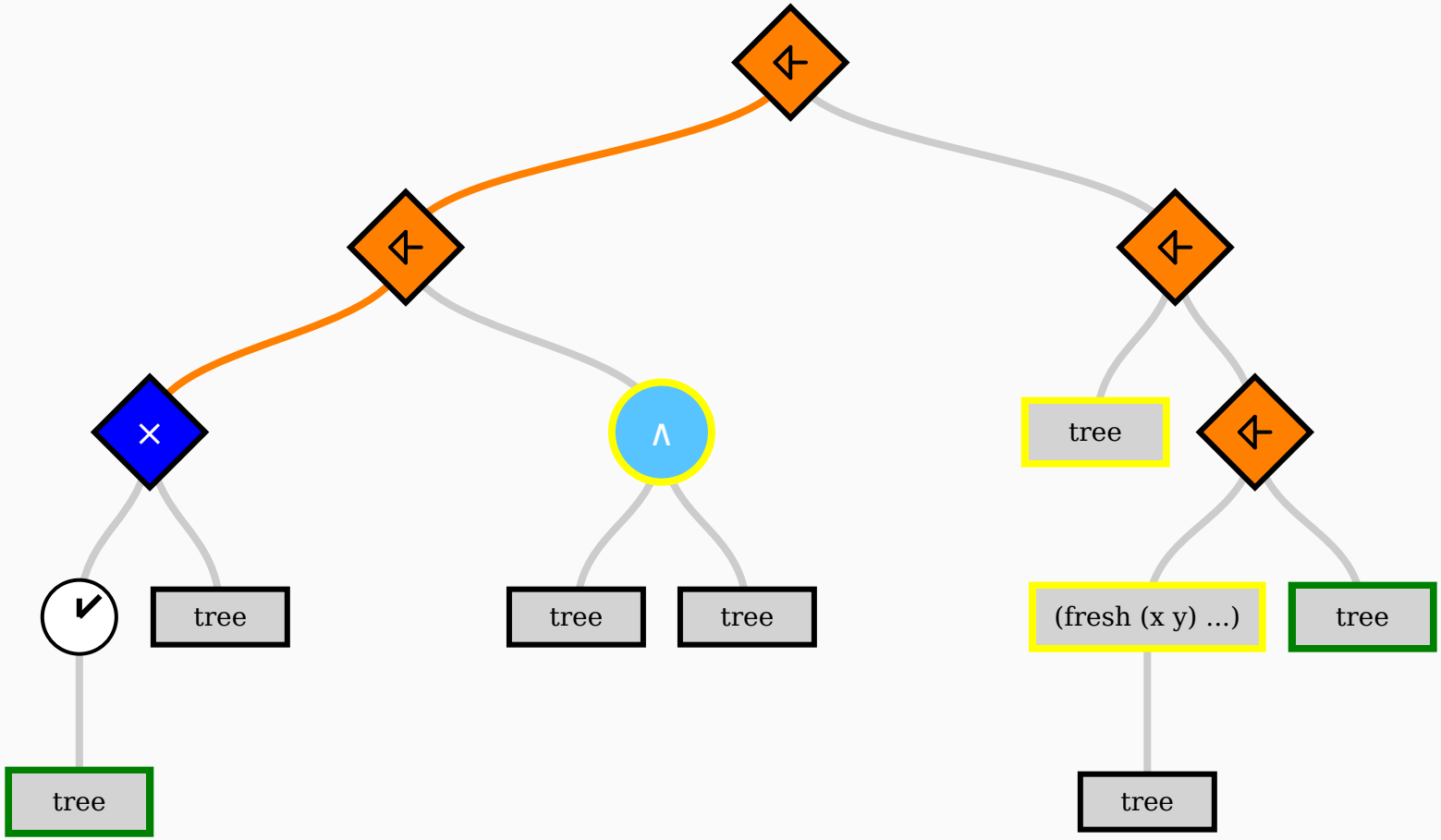

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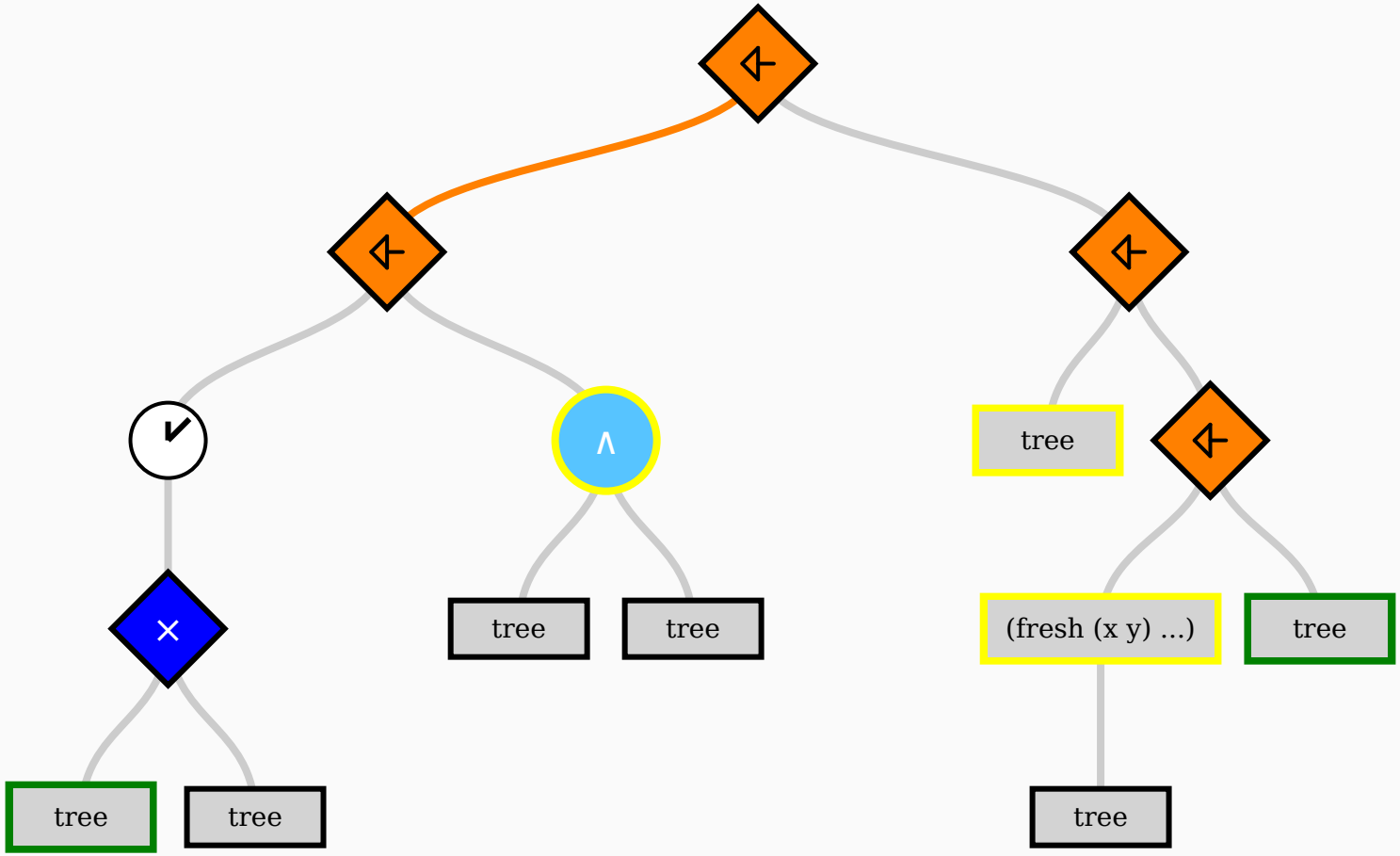
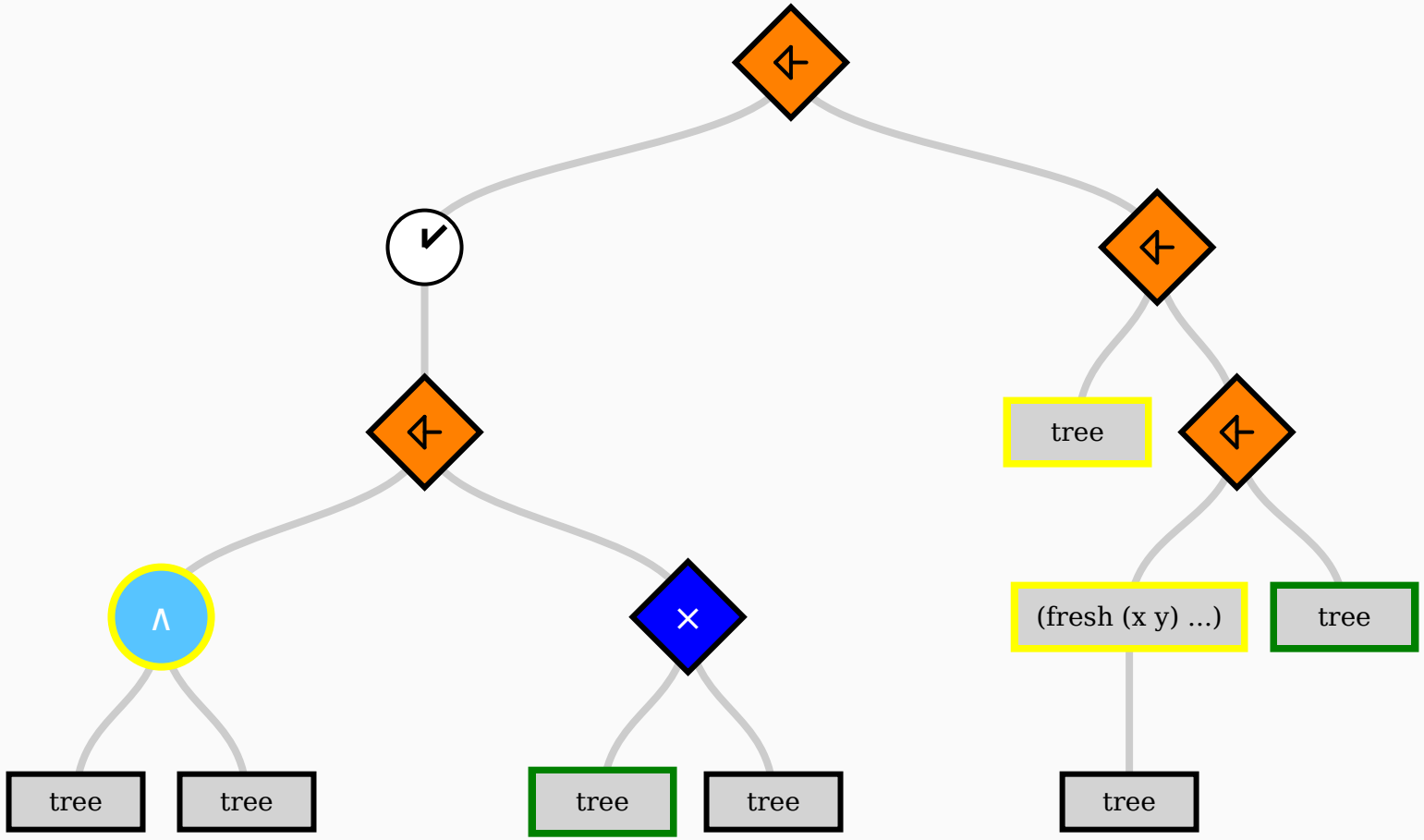
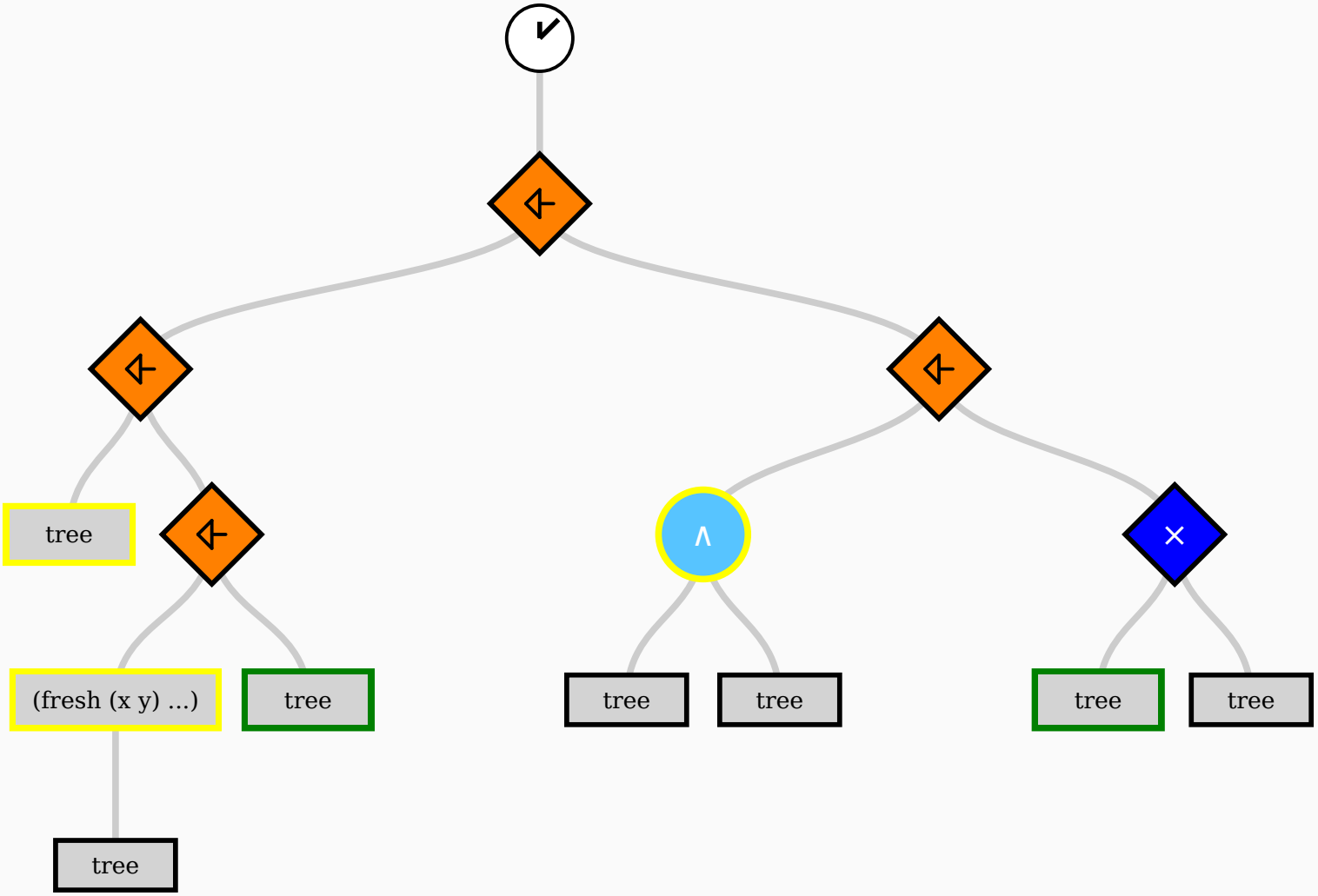


Old Interleaving



Old Interleaving



Railway Model :: Right Disjunctions

$$\text{Search Trees} \in S ::= S \leftarrow S$$

$$\mid \texttt{delay}\ S$$

Railway Model :: Right Disjunctions

$$\text{Search Trees} \in S ::= S \leftarrow S$$

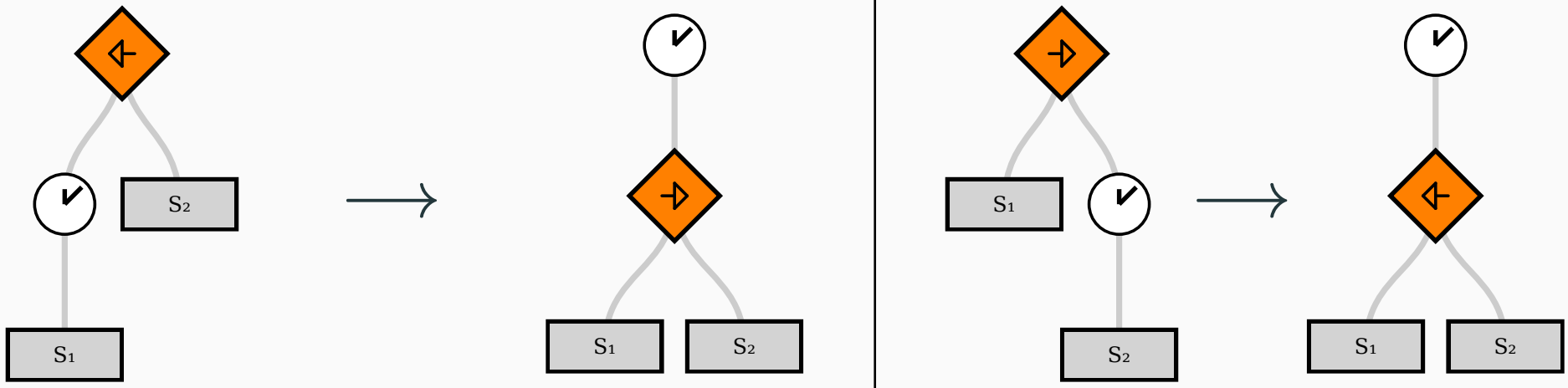
$$\mid S \rightarrow S$$

$$\mid \texttt{delay}\ S$$

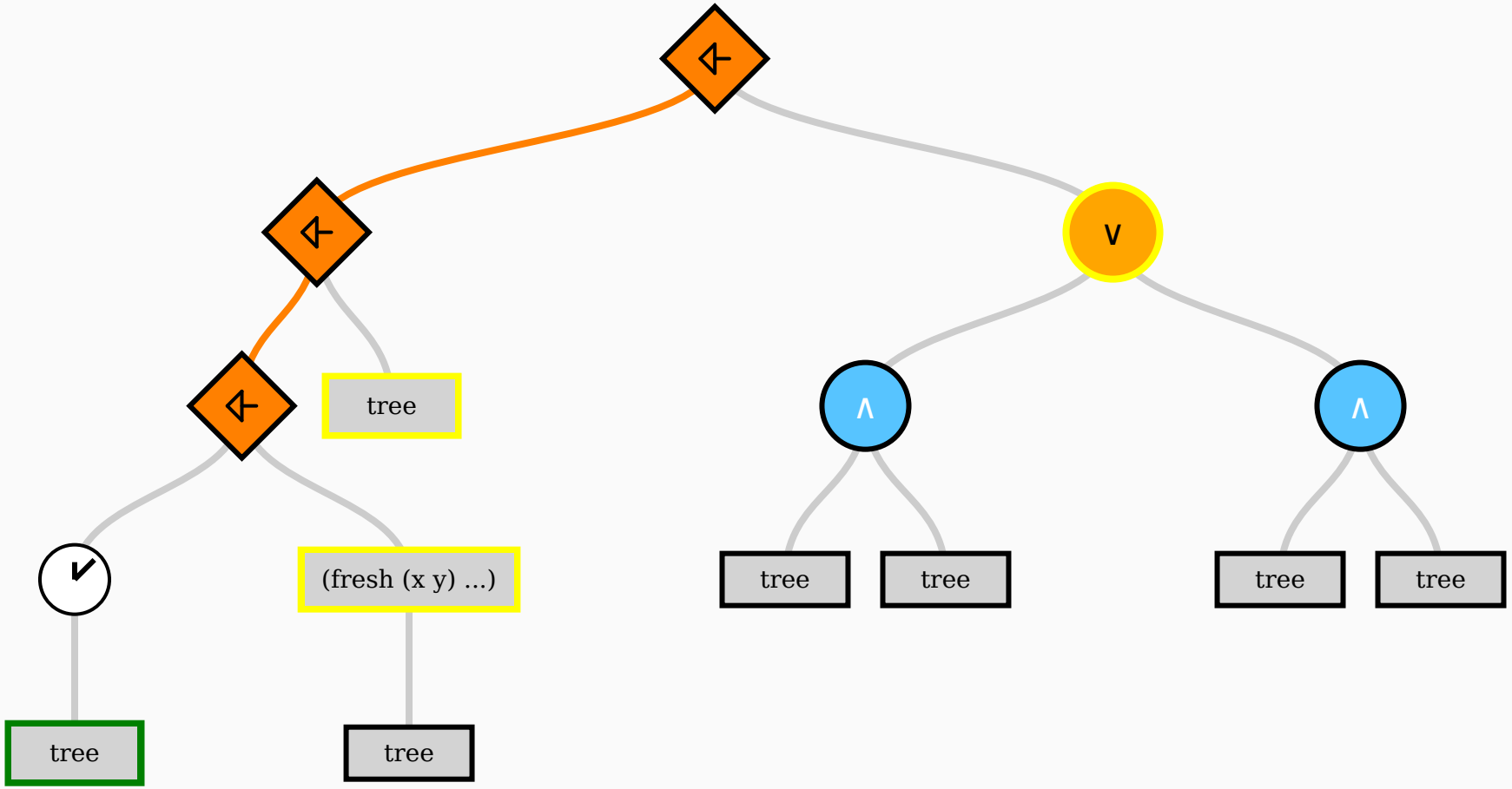
Railway Model :: Interleave Rules

$$E[(\texttt{delay}\ S_1) \leftarrow S_2] \longrightarrow E[\texttt{delay}\ (S_1 \rightarrow S_2)] \quad [\text{InterleaveLeft}]$$

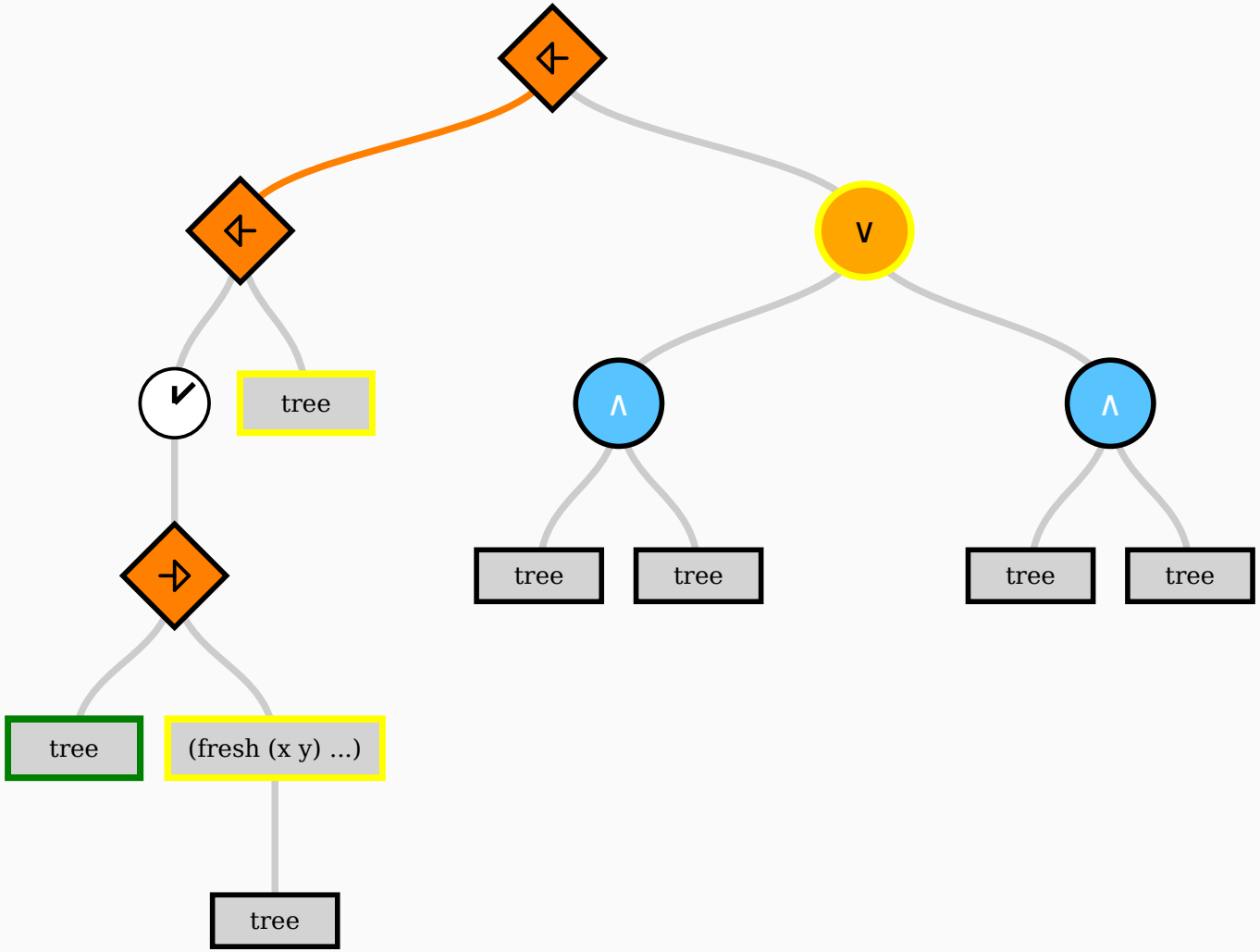
$$E[S_1 \rightarrow (\texttt{delay}\ S_2)] \longrightarrow E[\texttt{delay}\ (S_1 \leftarrow S_2)] \ [\text{InterleaveRight}]$$



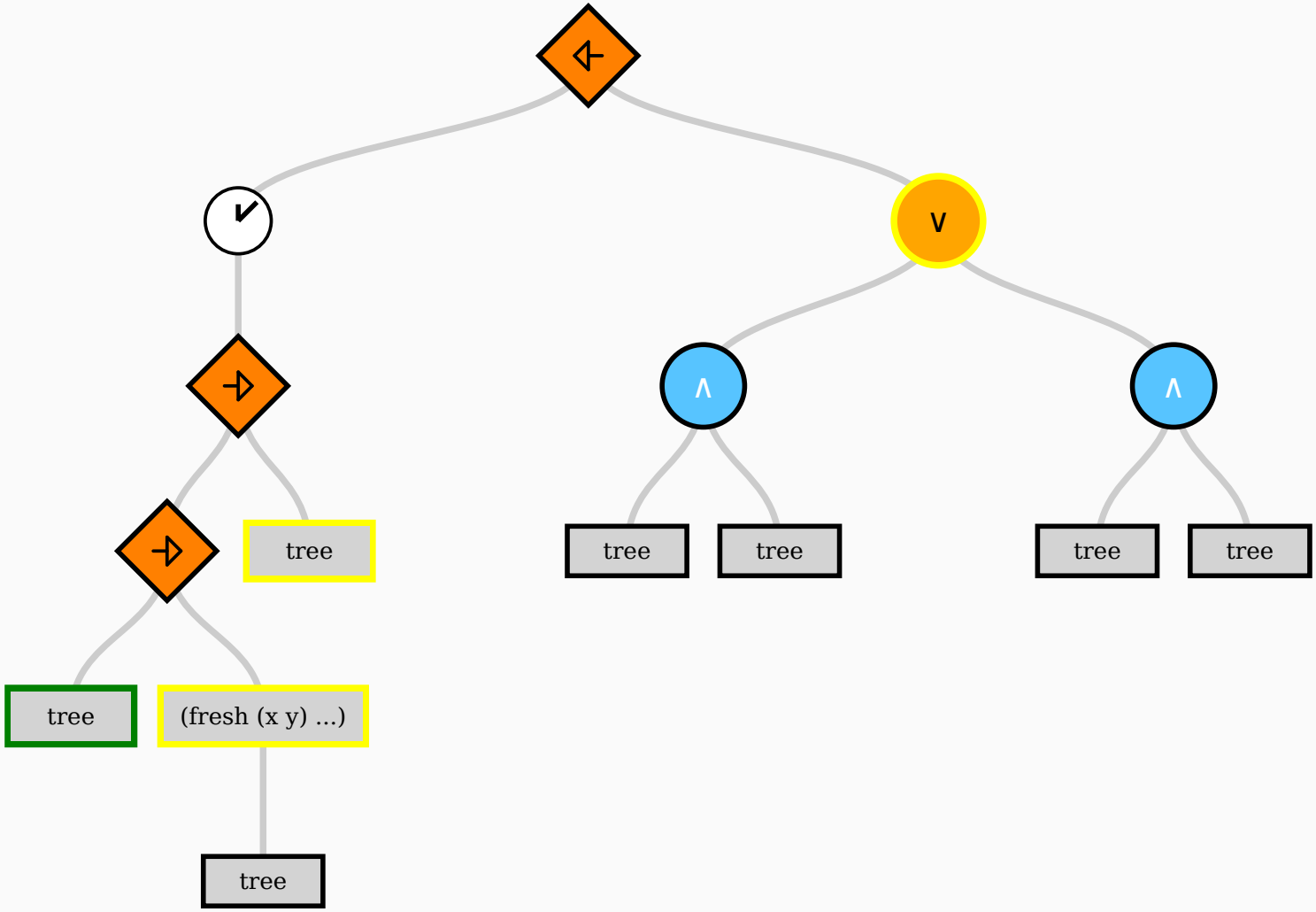
Railway Model



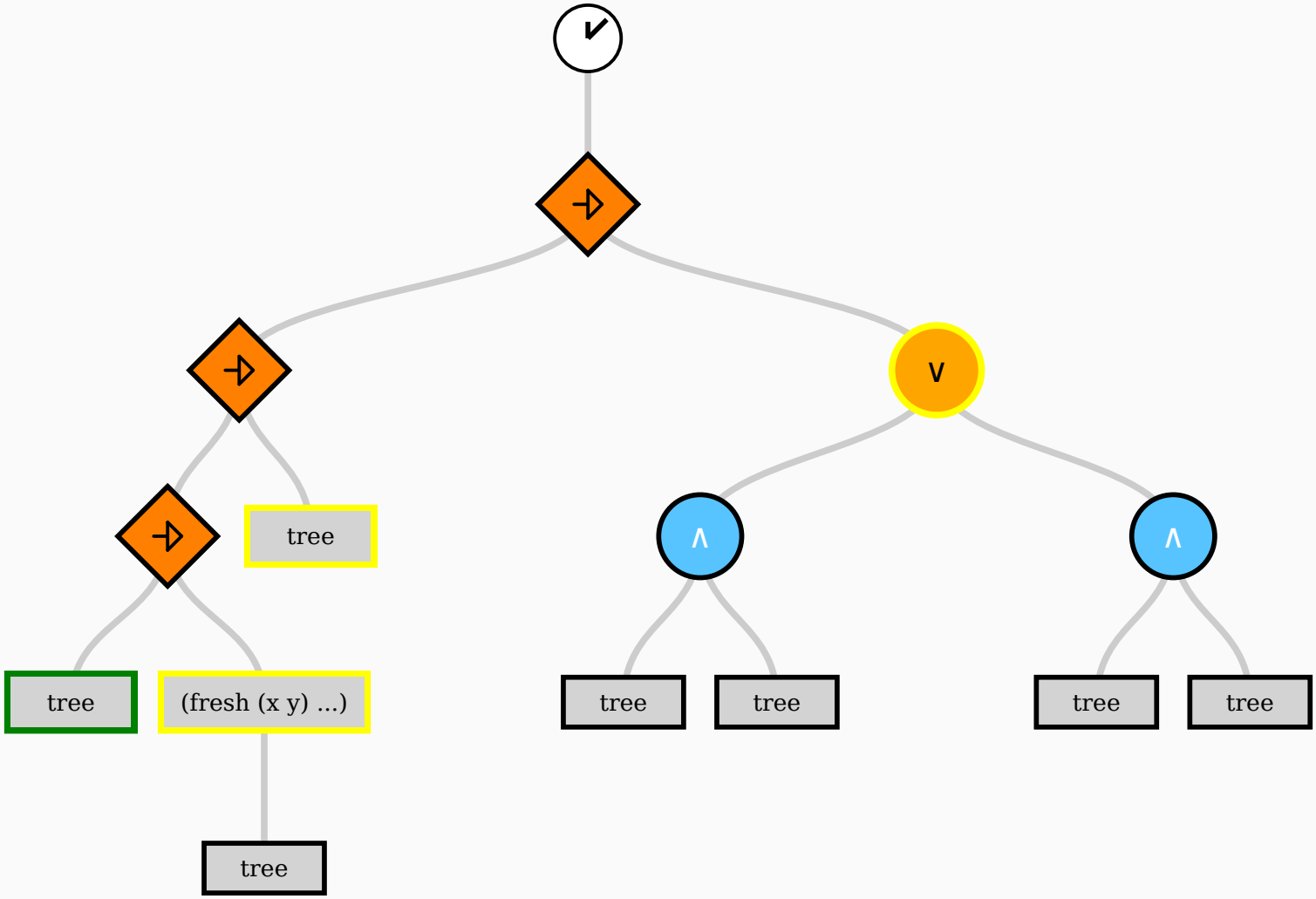
Railway Model



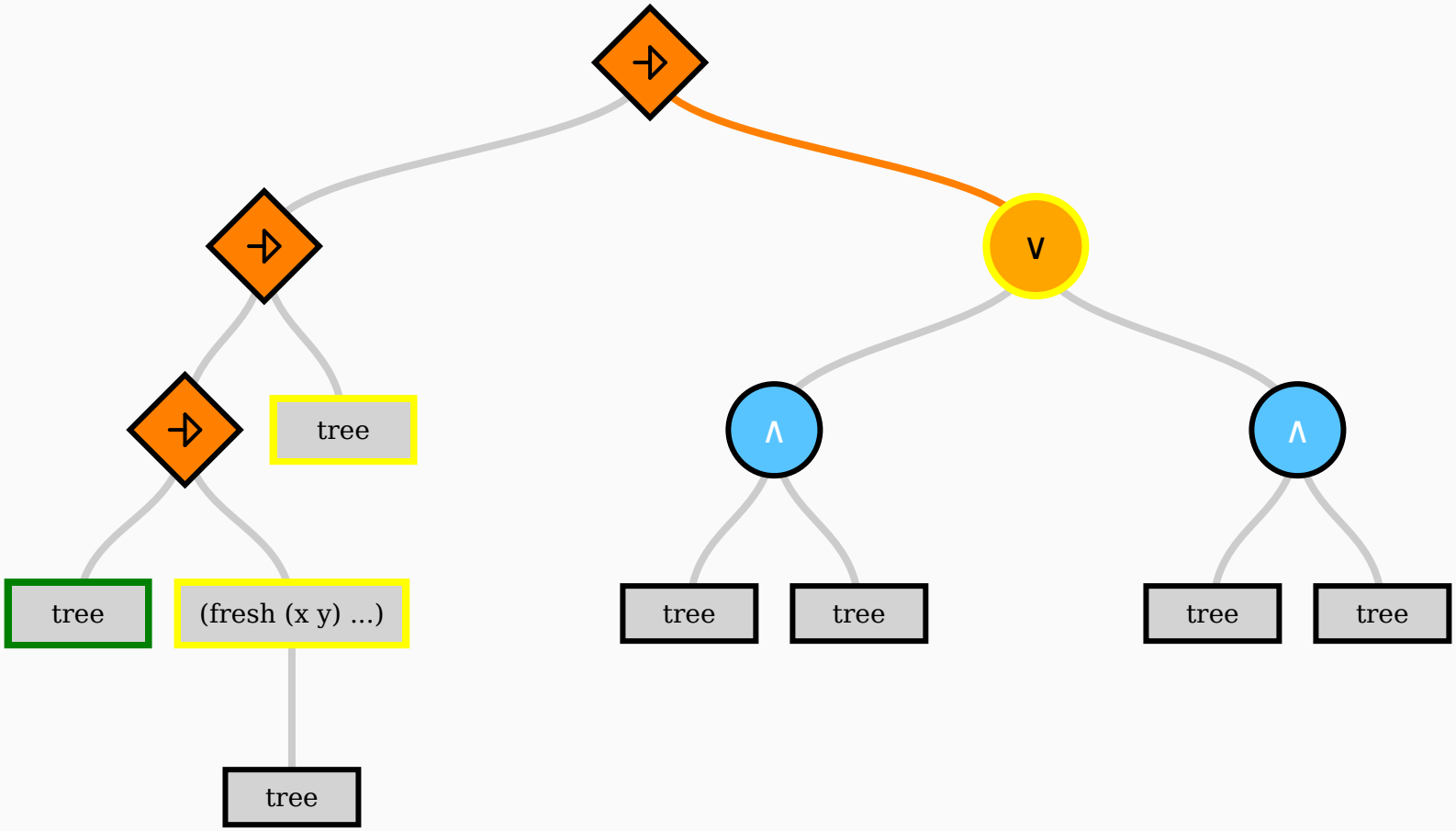
Railway Model



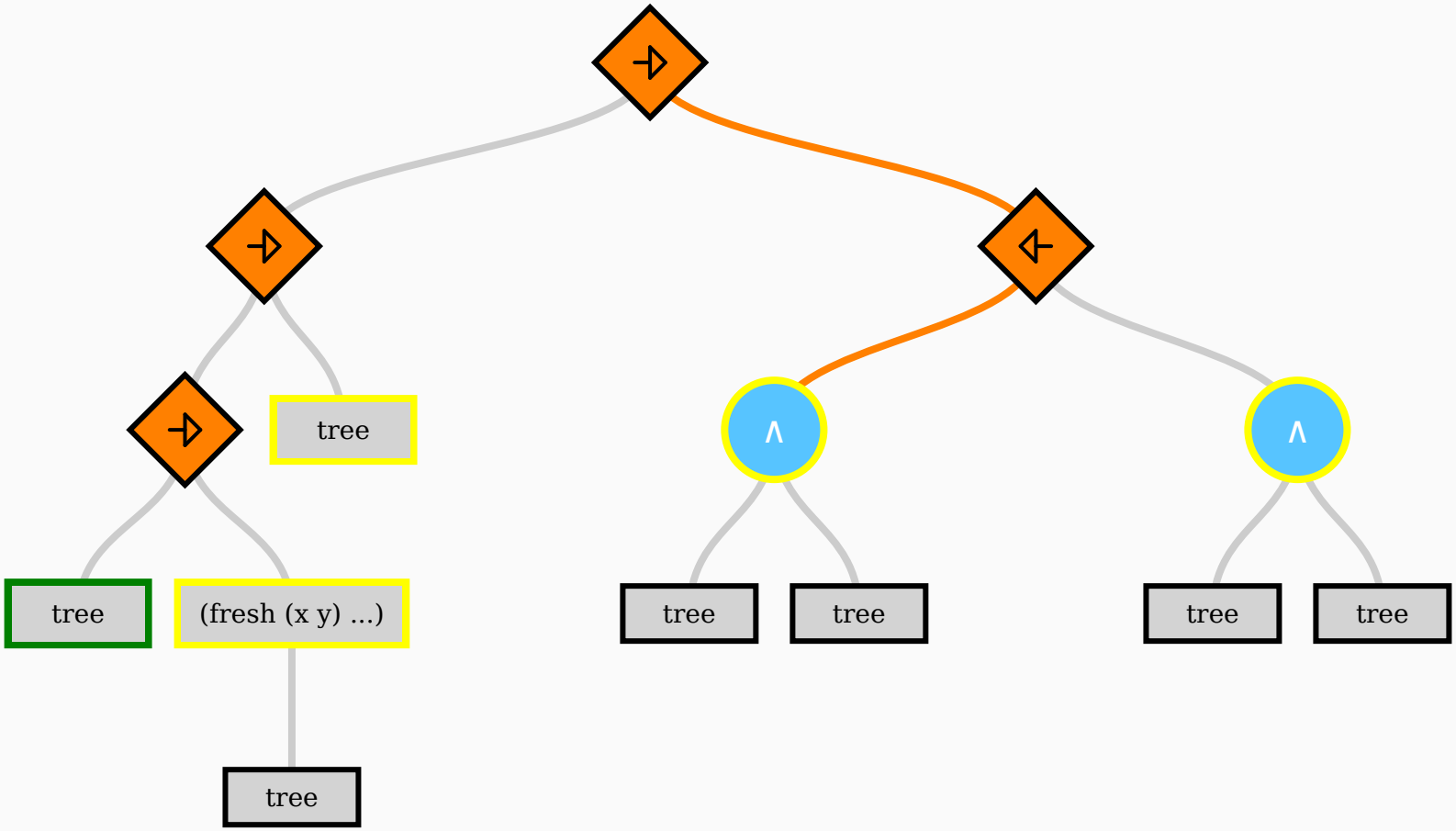
Railway Model



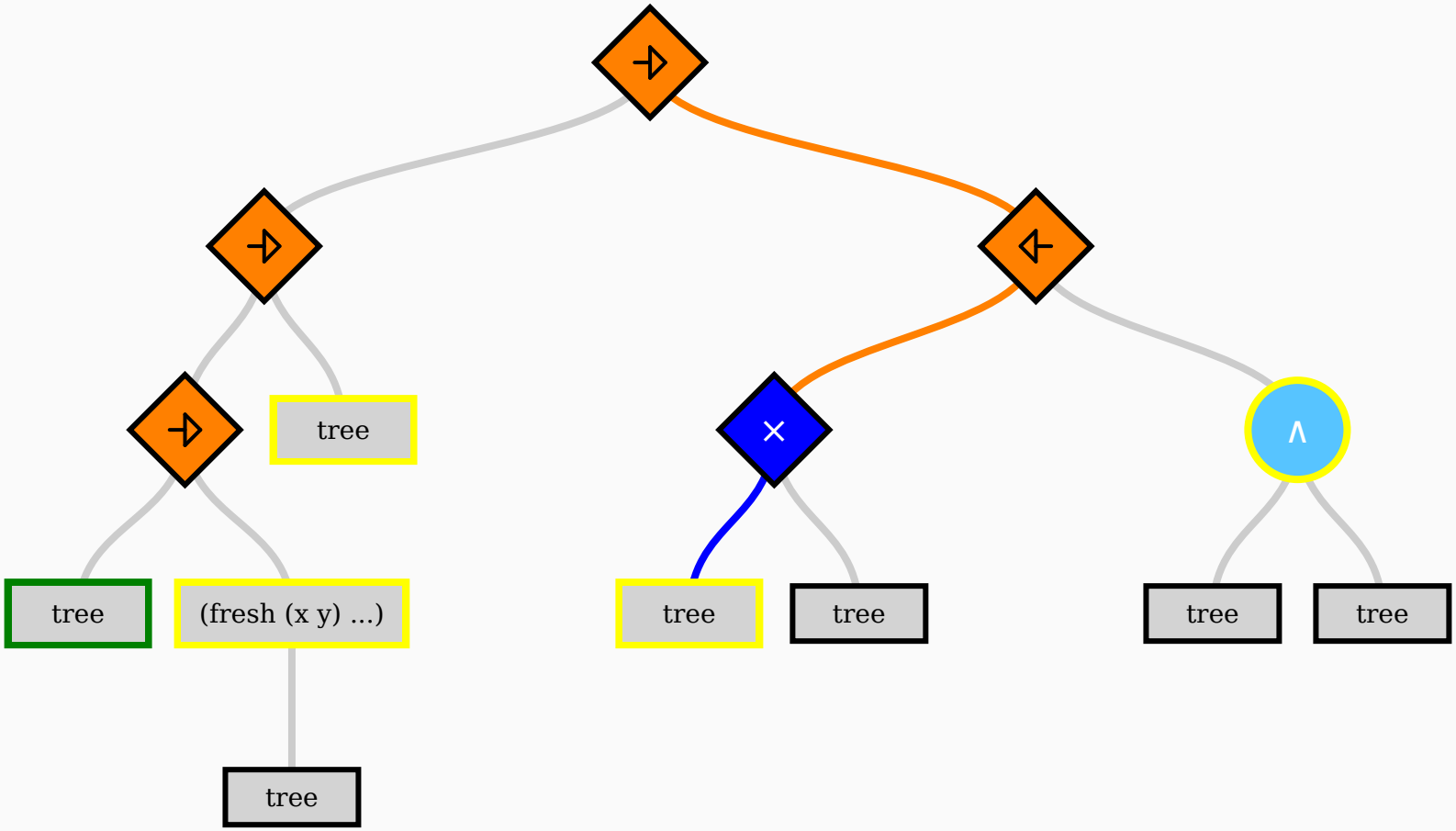
Railway Model



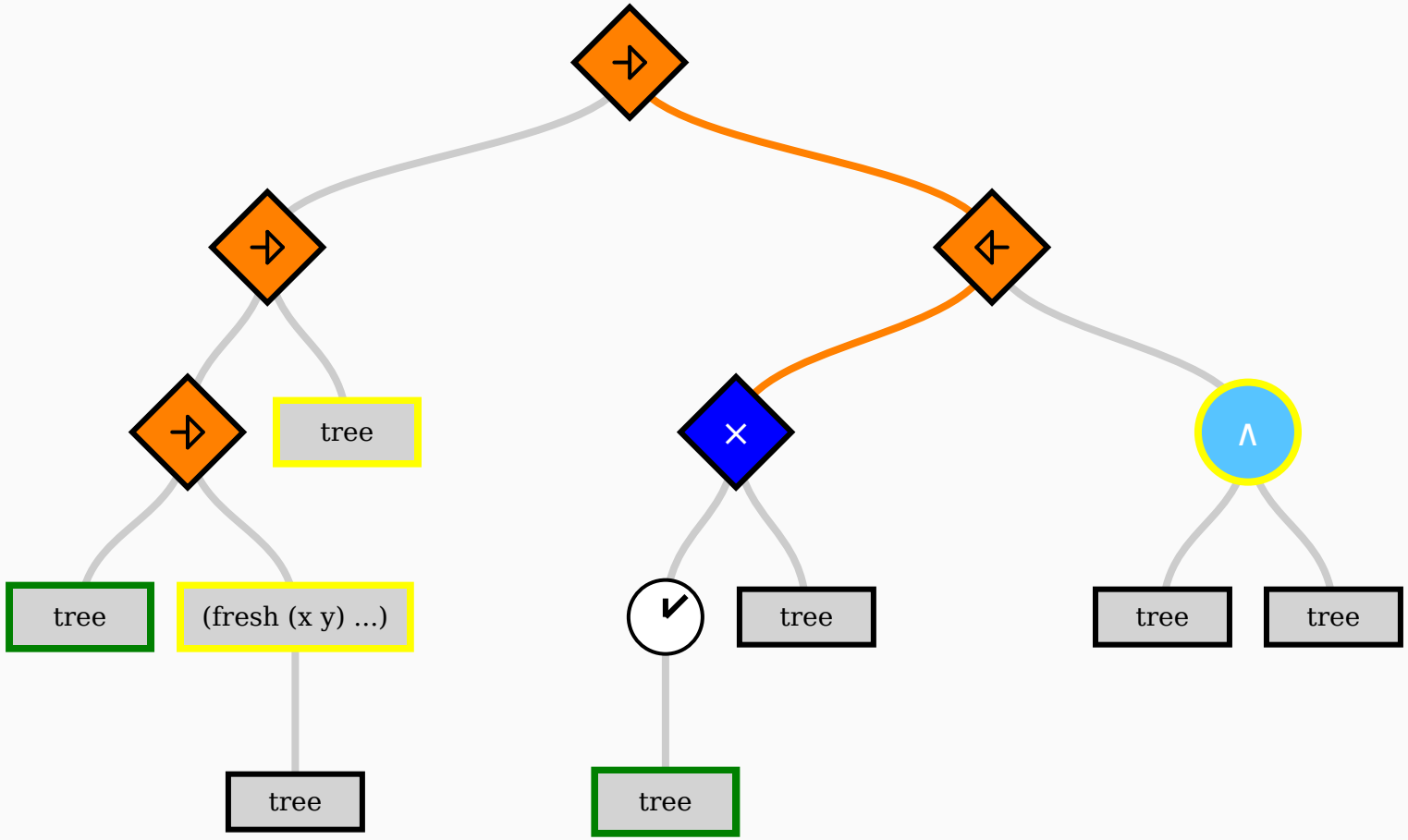
Railway Model



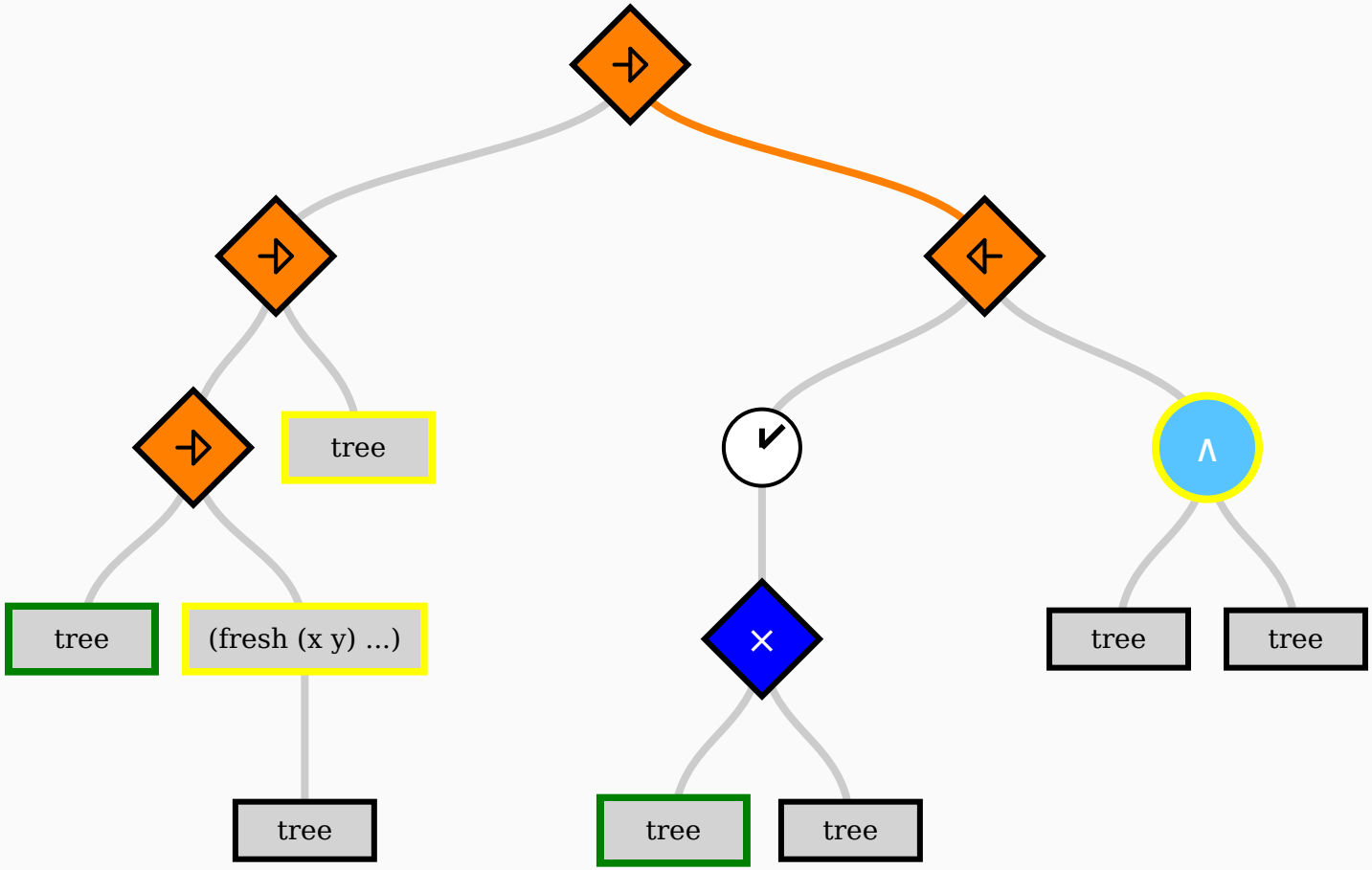
Railway Model



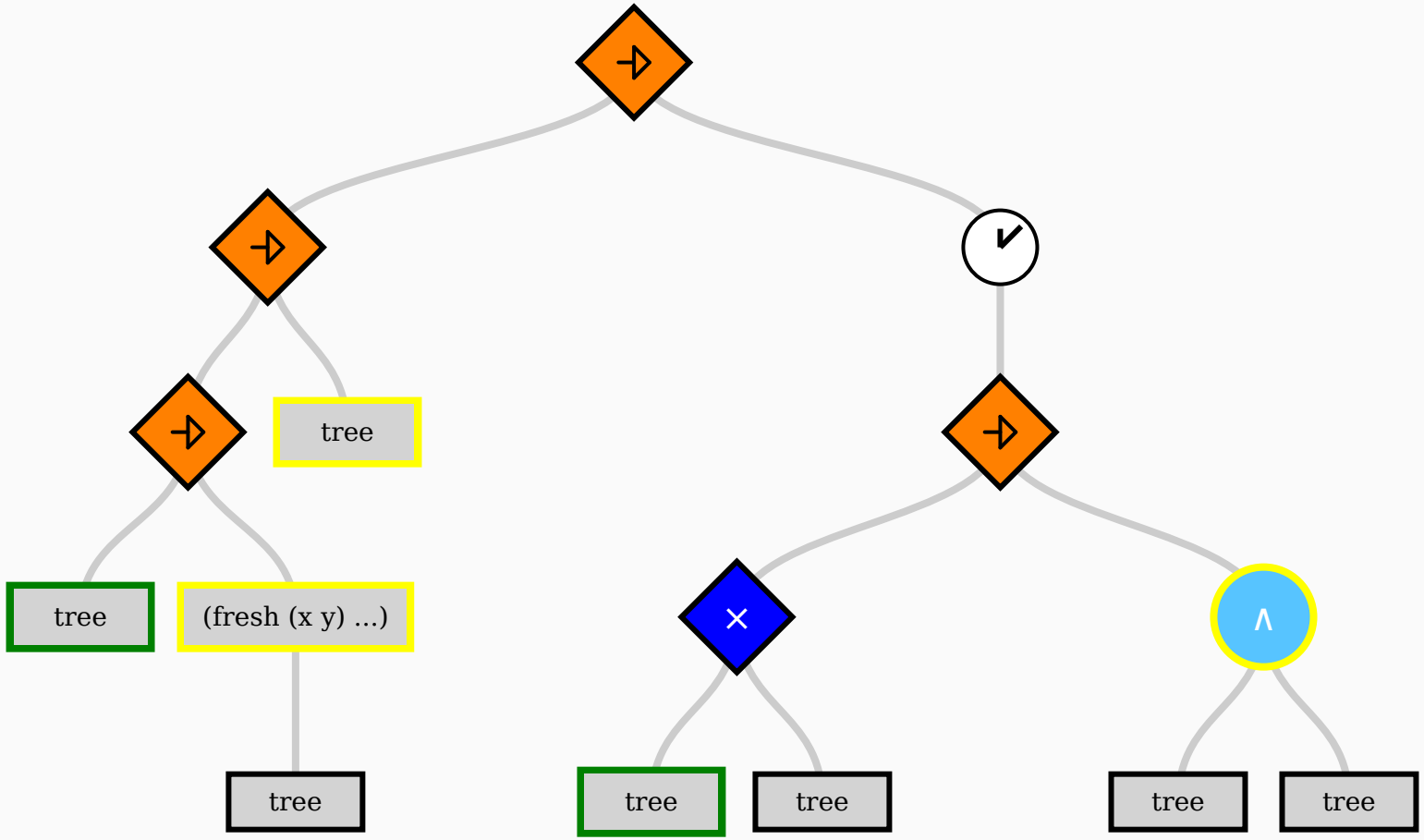
Railway Model



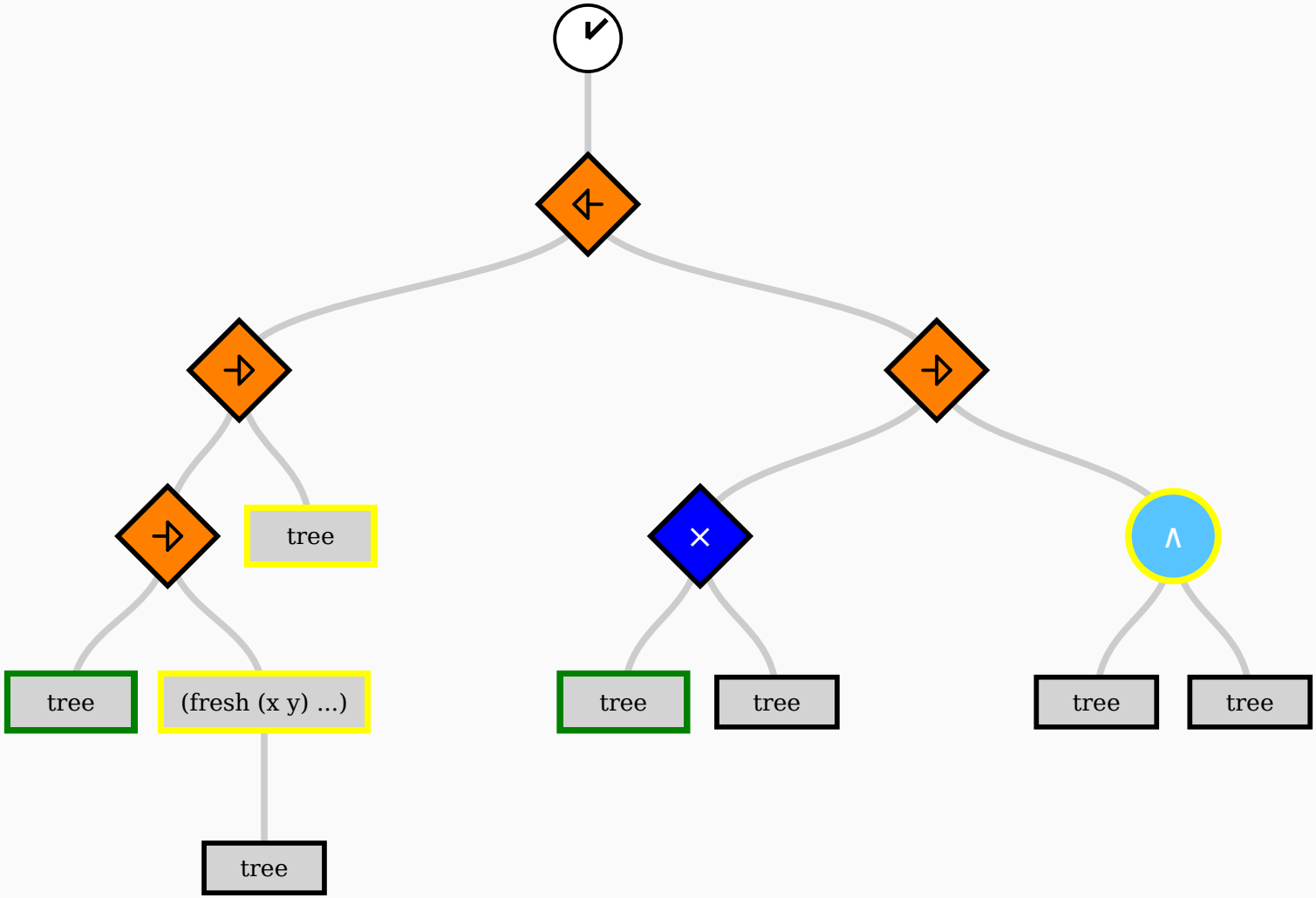
Railway Model



Railway Model



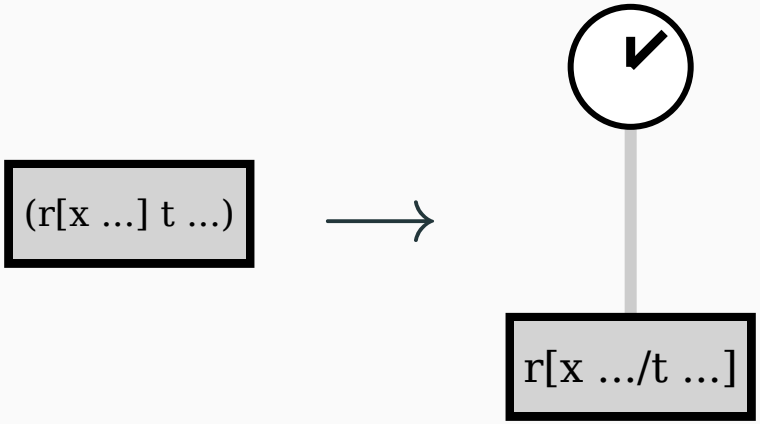
Railway Model



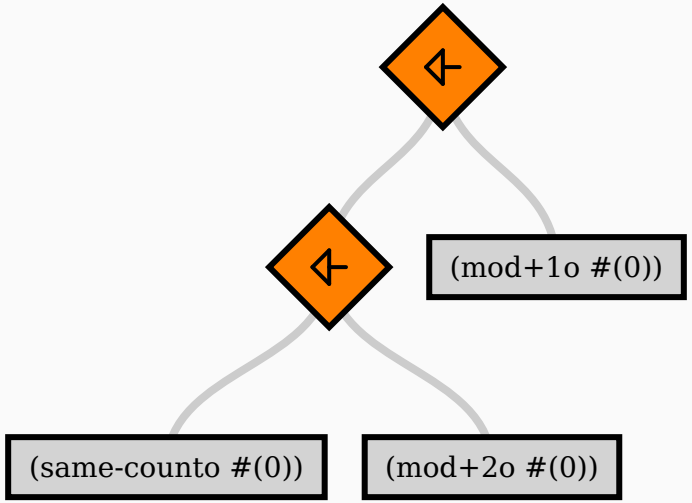
Lazy Relation Expansion

Old Delay

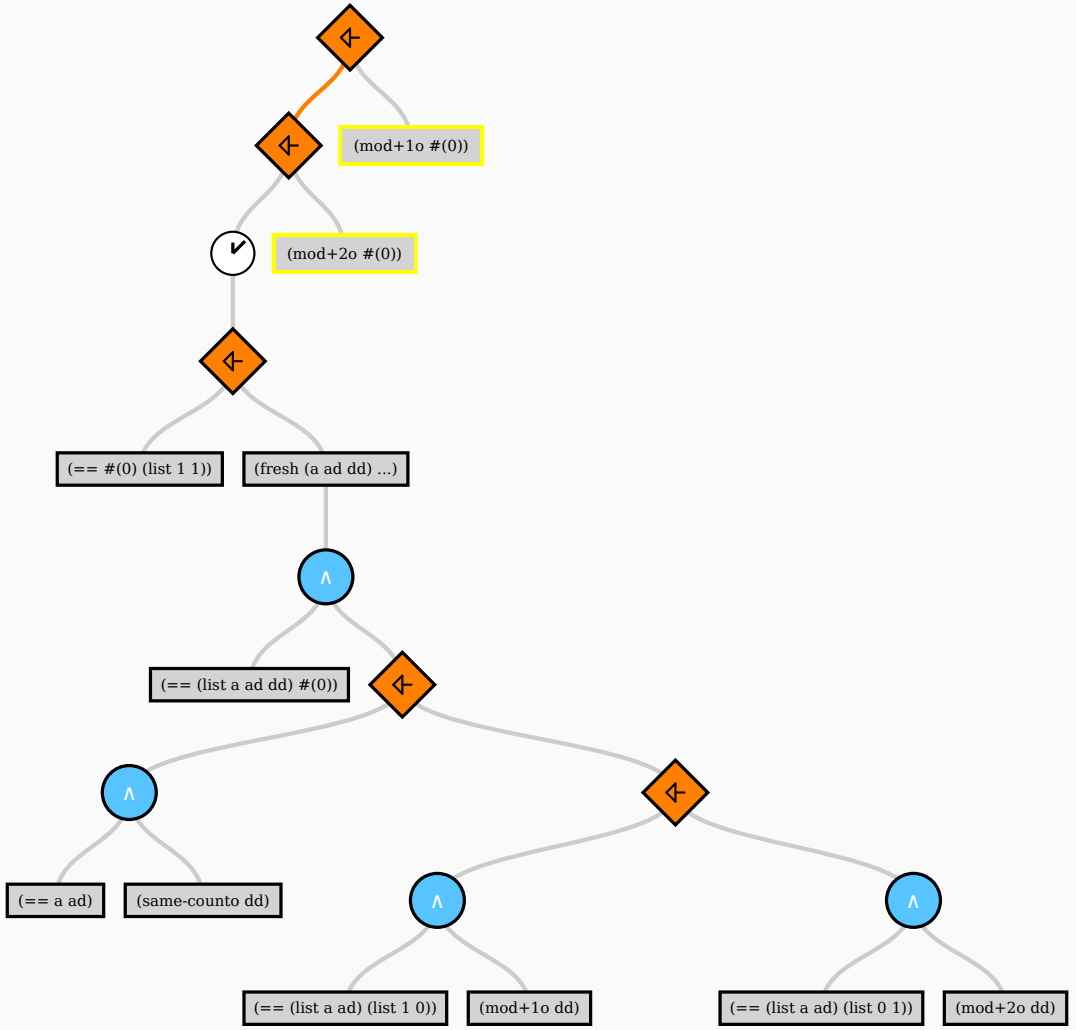
$$E[(r[x...]\ t...)\ \sigma] \longrightarrow E[\texttt{delay}\ \ r[x.../t...]\ \sigma]\ \text{[Delay (Old)]}$$



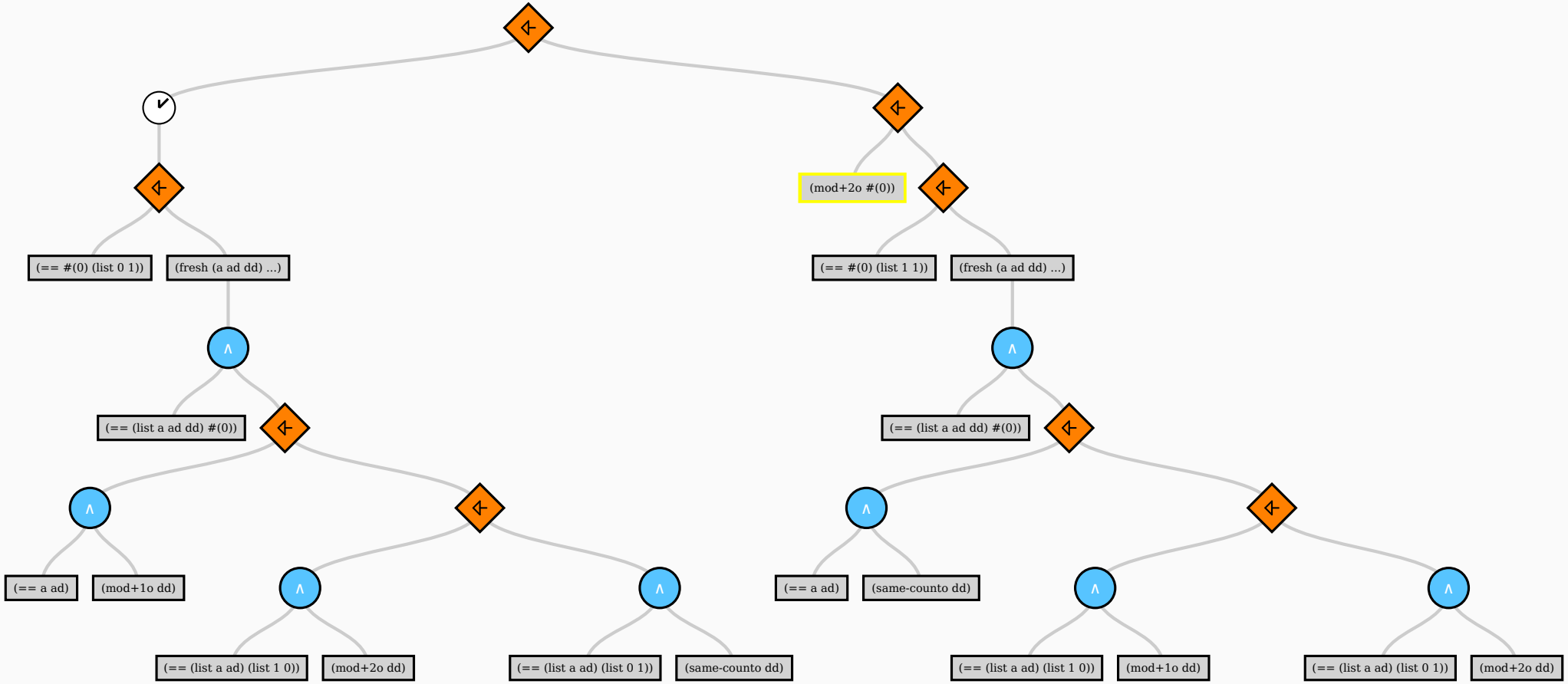
Old Delay



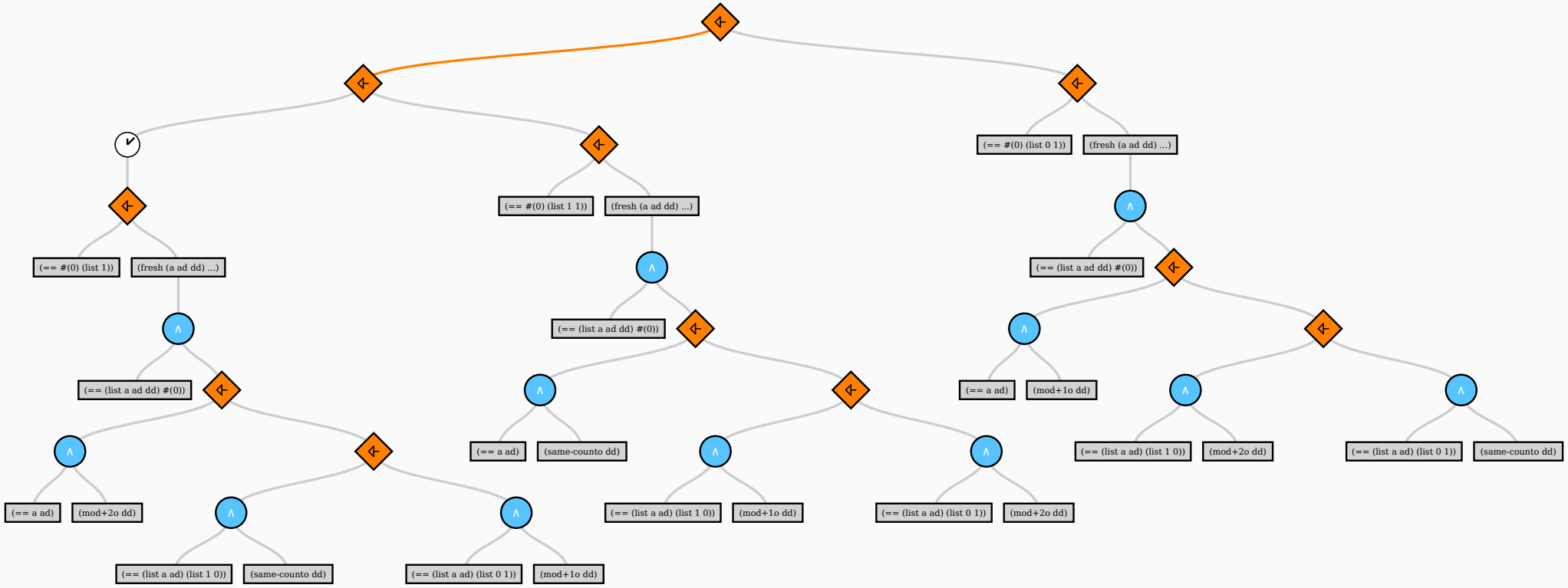
Old Delay


(mod+1o #(0))
(mod+2o #(0))
(== #(0) (list 1 1))
(fresh (a ad dd) ...)
∧
(== (list a ad dd) #(0))
∧
(== a ad)
(same-counto dd)
∧
(== (list a ad) (list 1 0))
(mod+1o dd)
∧
(== (list a ad) (list 0 1))
(mod+2o dd)

Old Delay

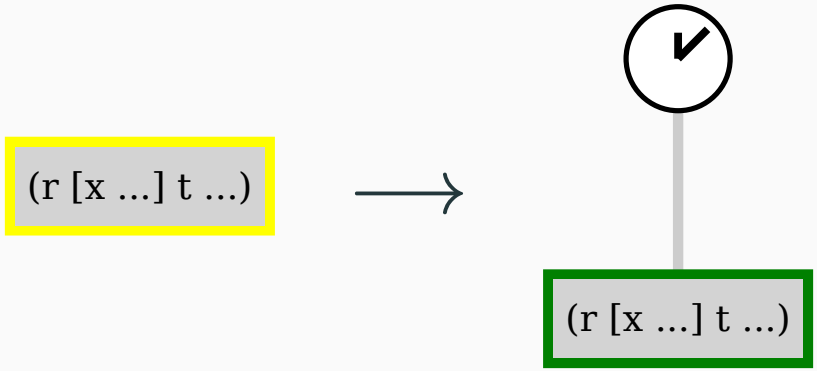

(== #(0) (list 0 1))
(fresh (a ad dd) ...)
(== (list a ad dd) #(0))
(== a ad)
(mod+1o dd)
(== (list a ad) (list 1 0))
(mod+2o dd)
(== (list a ad) (list 0 1))
(same-counto dd)
(mod+2o #(0))
(== #(0) (list 1 1))
(fresh (a ad dd) ...)
(== (list a ad dd) #(0))
(== a ad)
(same-counto dd)
(== (list a ad) (list 1 0))
(mod+1o dd)
(== (list a ad) (list 0 1))
(mod+2o dd)

Old Delay


(== #(0) (list 1))
(fresh (a ad dd) ...)
(== (list a ad dd) #(0))
(== a ad)
(mod+2o dd)
(== (list a ad) (list 1 0))
(same-counto dd)
(== (list a ad) (list 0 1))
(mod+1o dd)
(== #(0) (list 1 1))
(fresh (a ad dd) ...)
(== (list a ad dd) #(0))
(== a ad)
(same-counto dd)
(== (list a ad) (list 1 0))
(mod+1o dd)
(== #(0) (list 0 1))
(fresh (a ad dd) ...)
(== (list a ad dd) #(0))
(== a ad)
(mod+1o dd)
(== (list a ad) (list 1 0))
(mod+2o dd)
(== (list a ad) (list 0 1))
(same-counto dd)
(== (list a ad) (list 0 1))
(mod+2o dd)

Lazy Relation Expansion

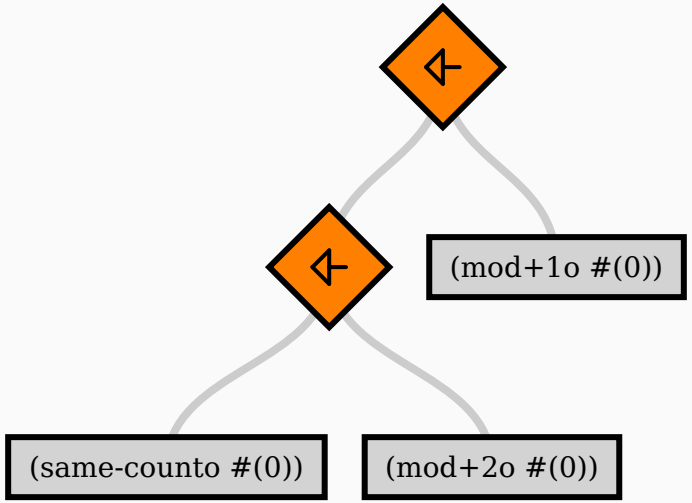
$$E[(r[x...]\ t...)\ \sigma] \longrightarrow E[\texttt{delay}\ (\texttt{go}\ (r[x...]\ t...)\ \sigma)]\ [\text{Delay}]$$



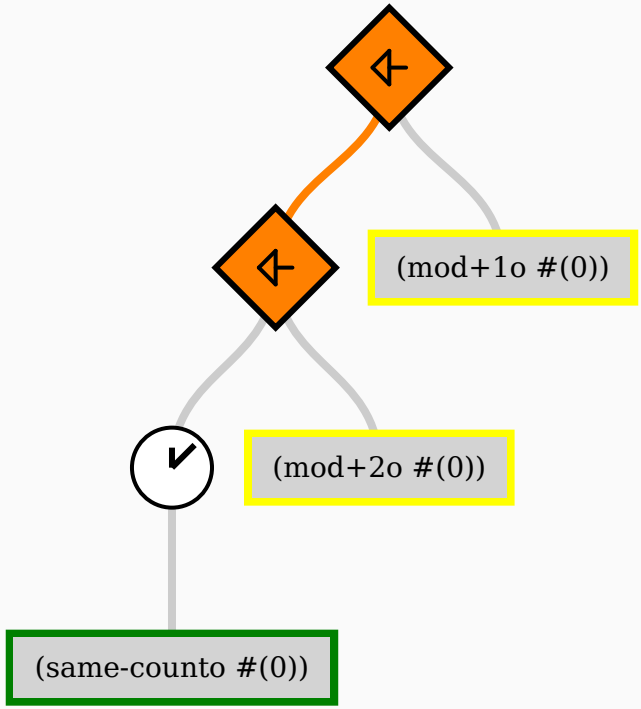
Lazy Relation Expansion

$$E[\texttt{go}\ (r[x\ldots]\ t\ldots)\ \sigma] \longrightarrow E[r[x\ldots/t\ldots]\ \sigma]\ \text{[Expand]}$$

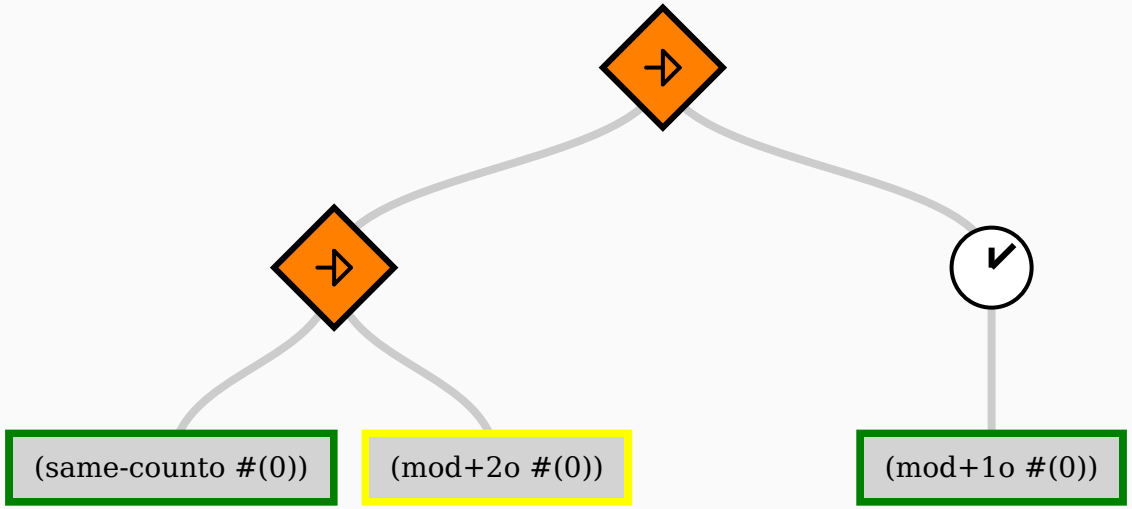
Lazy Relation Expansion



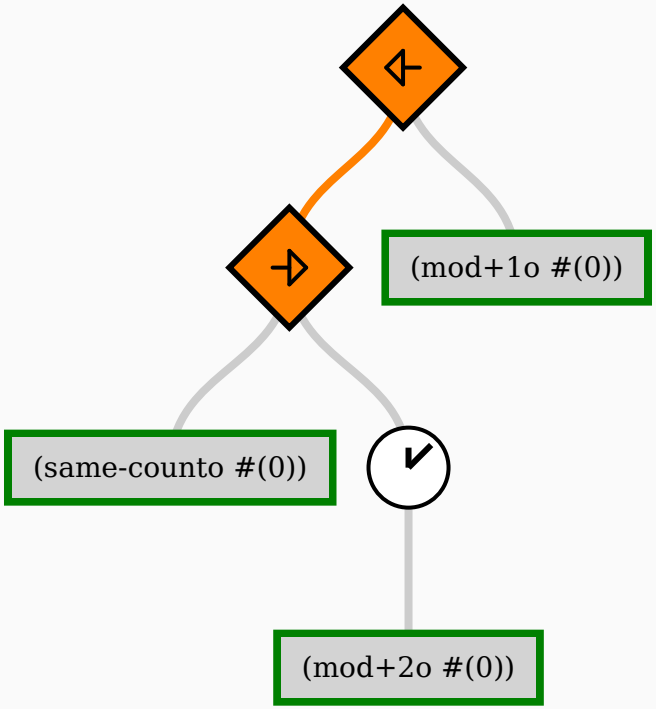
Lazy Relation Expansion



Lazy Relation Expansion



Lazy Relation Expansion

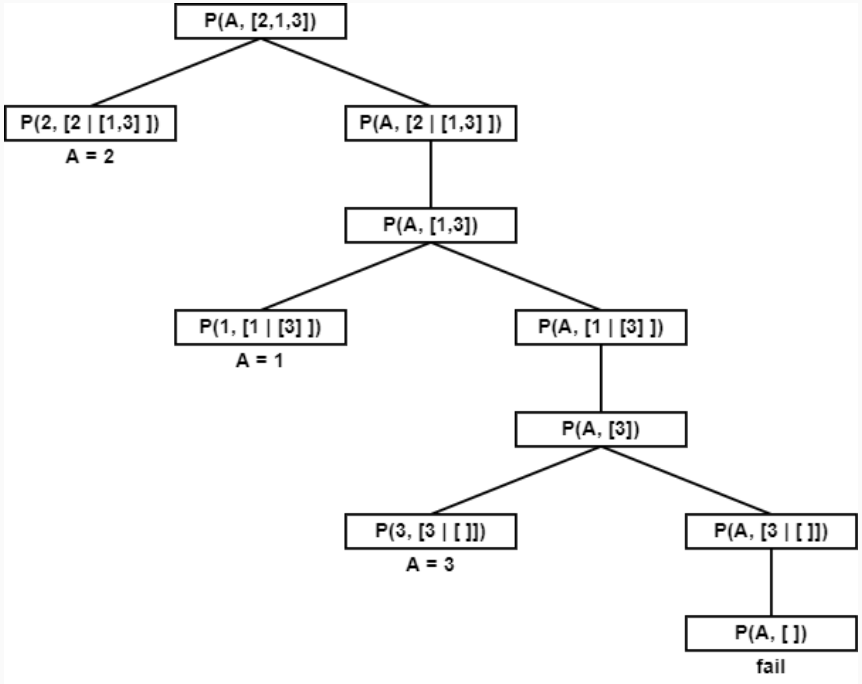


Demo

Existing Logic Programming Tooling (Prolog)

```
[trace]  ?- member(A, [2, 1, 3]).

Call : (12) member(_41716, [2, 1, 3])
Exit : (12) member(2, [2, 1, 3])

A = 2 ;

Redo : (12) member(_41716, [2, 1, 3])
Call : (13) member(_41716, [1, 3])
Exit : (13) member(1, [1, 3])
Exit : (12) member(1, [2, 1, 3])

A = 1 ;

Redo : (13) member(_41716, [1, 3])
Call : (14) member(_41716, [3])
Exit : (14) member(3, [3])
Exit : (13) member(3, [1, 3])
Exit : (12) member(3, [2, 1, 3])

A = 3 ;

Redo : (14) member(_41716, [3])
Call : (15) member(_41716, [])
Fail : (15) member(_41716, [])
Fail : (14) member(_41716, [3])
Fail : (13) member(_41716, [1, 3])
Fail : (12) member(_41716, [2, 1, 3])

false.
```

Byrd Box



AND/OR Tree

Existing Logic Programming Tooling (miniKanren)

```
================================================================================
Current Depth: 2              Number of Choices: 2

| Choice 1:
|   x = (1)
|   y = (1 2 3)
|   No constraints

| Choice 2:
|   x = (1 1 . #s(var b 21))
|   y = #s(var y 16)
|   Constraints:
|   * (appendoh #s(var b 21) #s(var y 16) (1 2 3))

[h]elp, [u]ndo, or choice number>
```

Textual Interactive Stepper - Rosenblatt et al. 2019

Conclusion

Future Directions

1. **Composable Search Strategies / Schedulers**
2. **Comparative Semantics**
3. **Mechanical Verification**

Conclusion

- **Semantics:** We define a fine-grained, deterministic small-step operational semantics making miniKanren's interleaving strategy explicit.
- **Visualizer:** A step-by-step tool exposes search control, interleaving choices, and suspended goals.

github.com/brysenPfingsten/mk-redex-viz

minikanrenredex-prod.shu.edu

Thank You!